THE ANALYSIS OF SOLUTION OF NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

FANIRAN TAYE SAMUEL
Assistant Lecturer,
Department of Computer Science,
Lead City University, Ibadan,
Nigeria.
Email : tayefaniran@yahoo.com

Abstract: Numerous known conditions, implying that a nonlinear differential equation u'+f(t,u,u') = 0 has solutions such that $\lim_{t \to \infty} \frac{u(t)-at}{t} = 0$ are local, in that the solutions are guaranteed to exist only for sufficiently large t. The aim of this paper is to review and present new theorems that ensure that solutions exist globally and exhibit "linear-like" behavior at infinity. An example illustrating the relevance of the theorem is discussed. Function spaces are introduced which are used for the proof of the main results. Their completeness is established and compactness criteria for the subsets of these spaces are also proved.

As asymptotic analysis is a method of describing limiting behavior which has applications across science and which is also a key tool for exploring the ordinary and partial differential equations that arise in the mathematical modeling of real-world phenomenon, it is important to investigate the future behavior of solutions of the differential equations.

Keywords: Prescribed asymptotic behavior; non-oscillatory solutions; nonlinear second-order differential equation; fixed point theory; linear-like behavior at infinity.

1. Introduction

For a nonlinear scalar equation:

$$u'' + f(t,u,u') = 0 \quad t \geq t_o \qquad (1)$$

to have solutions which exist on a given interval $[t_o,\infty)$ and which also have prescribed asymptotic behavior, the function f in Eq. (s1) must satisfy certain conditions on a given interval $[t_o, \infty)$. It is well known that in most cases, it is necessary to impose additional conditions on the differential equation in order to guarantee the global existence of solutions.

Eq. (1) plays an important role in mathematical modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neutrons which attracts constant interest of researchers.

The problem of establishing conditions under which all extendable solutions of eqt. (1) approach those of equation u'' = 0 is very closely related to the study of the existence of non-oscillatory solutions, having at least one zero, introduced for the first time in the paper by Nehari [9]. Nehari, Noussair and Swanson have also considered the question of global existence of solutions of the semilinear second order equation:

$$y'' + yg\,(t,y) = 0$$

and have given, for example, conditions which imply that this equation has a bounded positive solution, on a given interval $[t_o, \infty)$. This paper gives a review of the paper by Octavian G. Mustafa, Yuri V. Rogovchenko [10].

2. *Function Spaces and their Completeness*.

Three function spaces which will be used in the proofs of the main results shall be dealt with in this section. The first one denoted by A (t_o), is the set of all continuous real-valued functions u(t) on [t_o, ∞) which have limit l_u at infinity with the usual for C operations on functions and the space A (t_o) is complete if endowed with the usual sup-norm.

$$\|u\| = \sup_{t \geq t_o} |u(t)|$$

The second space, denoted by V(t_o), is the set of all continuously differentiable real-valued functions y(t) on [t_o, + ∞) with the property that their derivatives u'(t) have a limit a_u at infinity. It is also endowed with the usual for C' operations on functions.

Proposition 1: The space V(t_o) is complete under the norm.

$$\|u\| = \sup_{t \geq t_o} |u(t)| + \sup_{t \geq t_o} |u'(t)| \qquad \text{(a)}$$

Proof: Let $(u_n)_{n\geq1}$ be a Cauchy sequence in V(t_o). Then the sequence of derivatives $(u'_n)n\geq 1$ is a Cauchy sequence in A(t_o). We denote by v(t) its uniform limit as n→ +∞. Now by using the definition of Cauchy sequence, we shall show that for every T > t_o, the sequence $(u_n)_{n\geq1}$, is a Cauchy sequence in C([t_o, T], ℝ).

Let $(u_n)_{n\geq1}$ be a Cauchy sequence in V(t_o) for m, n≥ N_o (where N_o is a large natural number).

$\|u_n - u_m\| < \in, \quad \in > 0$

Now on C([to, T], ℝ)

$$\|u_n - u_m\|c = \sup_{t \in [t_o, T]} |u_n(t) - u_m(t)| = \sup_{t \in [t_o, T]} t\left|\frac{U_n(t) - U_m(t)}{t}\right|$$

$$\leq T \sup_{t \in [t_o, T]} \left|\frac{u_n(t) - u_m(t)}{t}\right|$$

$$\leq T \sup_{t \geq t_o} \left|\frac{u_n(t) - u_m(t)}{t}\right|$$

$$\leq T \left[\sup_{t \geq t_o} \left|\frac{U_n(t) - Um(t)}{t}\right| + \sup_{t \leq t_o} \left|u'_n(t) - u'_m(t)\right|\right]$$

$$= T \|u_n(t) - u_m(t)\| v(t_o)$$

$$< \quad T\in$$

Therefore, it has a local uniform limit as n→ + ∞, which we denote by u(t). On the other hand, using the integral representation of u_n

$$u_n(t) = u_n(t_0) + \int_{t_o}^{t} u'_n(s)ds, \qquad t \geq t_o$$

we deduce that u_n also has the local uniform limit u(t_o) + $\int_{t_o}^{t} v(s)ds, \quad t \geq t_o$

Therefore, uϵC' ([t_o, + ∞), ℝ) and u' = v. Since $(a_{u_n})_n \geq 1$ is a Cauchy sequence, after a straight forward computation, we conclude that $\lim_{t \to \infty} v(t)$ = a, where a = $\lim_{n \to +\infty} a_{u_n}$. Hence uϵv($t_o$) and completeness of the space V (t_o) is proved.

Remark: Note that formula (a) takes into account the fact that

$$\lim_{t \to \infty} \frac{u(t)}{t} = a \quad \text{if} \quad \lim_{t \to +\infty} u'(t) = a$$

$$\lim_{t \to +\infty} \frac{u(t)}{t} = \lim_{t \to +\infty} \frac{u(t)}{t} + \lim_{t \to \infty} \frac{k}{t} \quad \text{for} \quad k \epsilon \mathbb{R}$$

$$= \lim_{t \to \infty} \left(\frac{u(t)+K}{t}\right)$$

But

$$\lim_{t \to \infty} u'(t) = a$$

$$\lim_{t \to \infty} u(t) = ut + k$$

$$\lim_{t \to \infty} \frac{u(t)}{t} = a + \frac{k}{t}$$

$$\lim_{t \to \infty} \frac{u(t)}{t} = a$$

The third space, denoted by $W(t_0)$, is the set of all continuously differentiable real-valued functions u(t) on $[t_{0,} + \infty)$ having the property that $\lim_{t \to \infty} u'(t) - a_u = 0$

Now by integration, we have

$$\lim_{t \to \infty} u(t) - a_u t = b_u$$

3. *Compactness Criteria*

Definition: A space is said to be compact if every sequence in the space has a convergent subsequence in the space.

The compactness criterion for the subsets of the space $A(t_0)$ has been established by Avramescu [1].

Proposition 2: Assume that the subset $M \subset A(t_0)$ has the following properties:

(i) M is bounded, that is, there exists a constant.
L > 0 such that $|u(t)| \leq L$ for all $t \geq t_0$ and all $u \epsilon M$.

(ii) M is equicontinuous, that is, for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that
$|u(t_1) - u(t_2)| < \epsilon$ for all $t_1, t_2 \geq t_o$ satisfying $|t_1 - t_2| < \delta(\epsilon)$ and all $u \epsilon M$,

(iii) M is equiconvergent, that is, for every $\epsilon > 0$, there exists a $Q(\epsilon) > t_0$ such that $|u(t) - L_u| < \epsilon$ for all $t \geq Q(\epsilon)$ and all $u \epsilon M$.

Then, the subset M is relatively compact in $A(t_0)$. Conversely, if the subset M is relatively compact, then conditions (i), (ii), and (iii) are satisfied.

Proposition 3: Assume that the subset $M \subset V(t_0)$ has the following properties:

(i) There exists a constant L>0 such that $|u'(t)| \leq L$ and $\frac{|u(t)|}{t} \leq L$
for all $t \geq t_o$ and all $u \epsilon M$.

(ii) For every $\epsilon > 0$, $\exists$ a $\delta(\epsilon) > 0 \ni$
$|u'(t_1) - u'(t_2)| < \epsilon$ and $\left|\frac{u(t_1)}{t_1} - \frac{u(t_2)}{t_2}\right| < \epsilon$
for all $t_1, t_2 \geq t_0$ satisfying $|t_1 - t_2| < \delta(\epsilon)$ and for all $u \epsilon M$;

For every $\epsilon > 0$, $\exists$ a $Q(\epsilon) > t_0 \ni$
$|u'(t) - a_u| < \epsilon$ and $\left|\frac{u(t)}{t} - a_u\right| < \epsilon$

for all $t \geq Q(\epsilon)$ and all $u \epsilon M$.

Then the subset M is relatively compact in $V(t_0)$. Conversely, if the subset M is relatively compact, then conditions (i), (ii) and (iii) are satisfied.

Proof: Let $(u_n)_n \geq 1$ be a sequence in M. We shall show that it has a subsequence which converges in $V(t_0)$. It follows from $(i)_1$ $(ii)_1$ and $(iii)_1$ that the sequence of derivatives $(u'_n)_n \geq 1$ is relatively compact in $A(t_0)$. Therefore, there exists a subsequence of this sequence, also denoted $(u'_n)_n \geq 1$ which converges to a function v(t) in $A(t_o)$.

Consider now the corresponding subsequence $(u_n(t)/t)_{n} \geq 1$. According to $(i)_2$ $(ii)_2$ and $(iii)_2$, the subsequence $(u_n(t)/t)_n \geq 1$ is relatively compact in $A(t_0)$. Denote by $u(t)/t$, the limit of the subsequence $(u_n(t)/t)_n \geq 1$ as $n \to \infty$.

As in proof of proposition (1), we conclude that $u \epsilon C'([to, +\infty), \mathbb{R})$ and $u' = v$ and the proof is completed in a similar manner.

Lemma (1): Suppose that C is a convex subset of a normed linear space X ϶ $0 \subset C$. Let $\mu \epsilon [0,1]$. Assume that T: $c \to c$ is a completely continuous operator, then either there is an $x \epsilon C$ ϶ $x = \mu T(x)$.

Lemma (2): Let h: $[o, +\infty) \to [o, +\infty)$ be a continuous function. Assume that the function $f \epsilon ([o,+\infty), \mathbb{R})$ is non-negative and satisfies either of the following inequalities:

$$f(t) \leq k + \int_{t_o}^{t} h(s) g(f(s)) ds, \qquad t \geq t_o \qquad (a)$$

$$\text{or } f(t) \leq k + \int_{t_o}^{T} h(s) g(f(s)) ds, \qquad t_o + \leq t \leq T \qquad (b)$$

where $k \geq t_o$ is a real constant and g(s) is a continuous, positive and non-decreasing function such that

$$G(+\infty) = +\infty \qquad (c)$$

where $G(t) = \int_{t_o}^{t} \frac{ds}{g(s)}$

then

(a) If (a) holds, one has for all $t \geq t_o$ that

$$f(t) \leq G^{-1}\left(G(k) + \int_{t_o}^{t} h(s) ds\right), \qquad (d)$$

(b) If (b) holds, one has for all $t_o \leq t \leq T$ that

$$f(t) \leq G^{-1}\left(G(k) + \int_{t}^{T} h(s) ds\right), \qquad (e)$$

(ii) Suppose that h(s) is integrable on $[t_o, +\infty]$, while f(t) has a finite limit as $t \to \infty$ and satisfies the following integral inequality:

$$f(t) \leq k + \int_{t}^{\infty} h(s) g(f(s)) ds, \qquad t \geq t_o$$

Then, one has

$$f(t) \leq G^{-1}\left(G(k) + \int_{t}^{\infty} h(s) ds\right), \qquad t \geq t_o \qquad - \quad (f)$$

Proposition 4: Assume that the subset $M \subset W(t_o)$ has the following properties:

(i) ∃ exists a constant $L > o$ ϶

$|u'(t)| \leq L$ and $|u(t) - a_u t| \leq L$

for all $t \geq t_o$ and all $u \epsilon M$.

(ii) For every $\epsilon > o$, $\exists$ a $\delta(\epsilon) > o \ni$

$|u'(t_1 - u'(t_2)| < \epsilon$ and $|u(t_1) - u(t_2) - a_u(t_1 - t_2)| < \epsilon$

for all $t_1, t_2 \geq t_o$ satisfying $|t_1 - t_2| < \delta(\epsilon)$ and all $u \epsilon M$,

(iii) for every $\epsilon > o$, $\exists$ a $Q(\epsilon) > t_o \ni$

$|u'(t) - a_u| < \epsilon$ and $|u(t) - a_u t - bu| < \epsilon$

for all $t \geq Q(\epsilon)$ and all $u \epsilon M$. Then the set M is relatively compact in $W(t_o)$. Conversely, if the subset M is relatively compact, then conditions (i), (ii) and (iii) are satisfied.

Proof: Let $(u_n)_n \geq 1$ be a sequence in M. It follows from (i), (ii), and (iii), that there exists a subsequence of the sequence of derivatives $(u'_n)_n \geq 1$, which converges in $A(t_o)$ to a function v(t). Furthermore, $l_v = a$, where $a = \lim_{n \to \infty} a_{u_n}$. It follows from $(i)_2$, $(ii)_2$ and $(iii)_3$ that the subsequence $(u_n(t) - a_{u_n} t)_n \geq 1$ is relatively compact in A (t_o). Hence there exists a subsequence $(u_n(t) - a_{u_n} t)_n \geq 1$ of this sequence which converges in $A(t_o)$ to a function w(t). Furthermore, Lw = b, where $b = \lim_{n \to \infty} bu_n$.

Define the function u(t) by u(t) = w(t) + at. A straightforward computation similar to that in the proof of proposition 1 yields that u_n has the local uniform limit u(t) as $n \to +\infty$. Furthermore, we have that $u \epsilon C'([t_o, +\infty), \mathbb{R})$ and $u' = v$.

Since for every $\epsilon > o$, $\exists$ a positive integer $n_\epsilon \ni$

$\left|u_n(t) - u(t) - t(a_{u_n} - a)\right| < \epsilon$

for all $t \geq t_o$ and $n \geq n_\epsilon$, we obtain that

$\left|\frac{u_n(t)}{t} - \frac{u(t)}{t}\right| < \epsilon + \left|a_{u_n} - a\right|$

for all $t \geq t_o$ and $n \geq n_\epsilon$. Consequently, the sequence $(u_n(t)/t)_n \geq 1$ converges in $A(t_o)$ to the function $u(t)/t$.

4. Main Results

The idea of the proofs of Theorems 1-4 is as follows. First we transform all the problems associated to Eq. (1) into correspondent integral equations. Then the relevant integral operators acting on $V(t_o)$, $W(t_o)$ are introduced and their properties are examined.

In order to apply either the Leray-Schauder alternative (Lemma 1) or the classical Schauder-Tikhonov theorem (see, for instance, [5], p. 22]), one has to show that the integral operator is completely continuous (compact). The proof of this fact is divided into two steps. First, we show that the integral operator is uniformly continuous on bounded subsets of the corresponding function space. Second, we demonstrate that the image of a bounded set under the action of the integral operator satisfies hypotheses (i) – (iii) of Propositions 3 and 4, and, therefore, is relatively compact in the corresponding function space.

Since the proof of Theorem 1 is the most elucidative, we focus on it providing all the details. For the remaining results, only the characteristic features will be prompted. But before then, we shall state the following lemma:

Lemma 3. Assume that for the set $M \subset C^1([a,b], \mathbb{R})$ the following conditions are satisfied:

(i) there exists a constant $L > 0$ such that

$|u'(t)| \leq L$

for all t ∈[a,b] and all u ∈M:

(ii) for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$|u'(t_1) - u'(t_2)| < \varepsilon$ and $|u(t_1) - u(t_2)| < \varepsilon$

For all t_1, t_2∈ [a,b] satisfying $|t_1 - t_2| < \delta(\varepsilon)$ and for all u ∈ M

Then, the set M is relatively compact in C^1 ([a,b), ℝ).

Theorem 1: Suppose that these conditions

$$|f(t,u,v)| \leq h(t)\left[P_1\left(\frac{|u|}{t}\right) + P_2(|v|)\right], \quad (1)$$

$$\int_{t_o}^{\infty} h(s)\, ds = H < +\infty \quad (2)$$

and G(+ ∞) = + ∞ are satisfied. Then for every aϵ ℝ, there exists a solution u(t) of Eq. (1) defined on [t_o, + ∞] which has the asymptotic representation u(t) = at + o(t) as t → + ∞.

Proof of Theorem 1 . *Step 1*: Fix a real number b > 0, and let λ ∈(0,1). It follows from (2) that there exists a $t_* = t_*(b,\lambda) > t_0$ such that

$$\int_{t_*}^{\infty} h(s)ds < \lambda\frac{b}{3(p_1(b)+p_2(b))} \quad (3)$$

Selecting real numbers u_0 and a so that

$$\frac{|u_0|}{t_*} + 2|a| < (1-\lambda)b.$$

Consider the set C = {u ∈ V(t_*): $\|u\| \leq b$} and define the operator $T:C \to C$ by the formula

$$(Tu)(t) = u_0 + a(t - t_*) + (t - t_*)\int_{t_*}^{\infty} f(s,u(s),u'(s))ds$$

$$- \int_{t_*}^{1}(t-s)f\big(s,u(s),u'(s)\big)ds, \quad t \geq t_*$$

It follows from (1) that for all $t \geq t_*$

$$|(Tu)'(t)| \leq |a| + \int_{t}^{\infty}|f(s,u(s).u'(s))|ds$$

$$\leq |a| + (p_1(b) + p_2(b))\int_{t_*}^{\infty} h(s)ds$$

and

$$\frac{|(Tu)(t)|}{t} \leq \frac{|u_0|}{t_*} + |a| + \int_{t_*}^{\infty}|f(s,u(s),u'(s))|\, ds + \int_{t_*}^{t}|f(s,u(s),u'(s)|ds.$$

$$\leq \frac{|u_0|}{t_*} + |a| + 2(p_1(b) + p_2(b))\int_{t_*}^{\infty} h(s)ds.$$

Consequently, we obtain the following estimate for the norm of the operator *T:*

$$\|Tu\| \leq \frac{|u_0|}{t_*} + 2|a| + 3(p_1(b))\int_{t_*}^{\infty} h(s)ds.$$

$$< (1-\lambda)b + \lambda b = b,$$

which implies that T is well defined.

Next, we have to prove that the operator T is continuous and the set TC is relatively compact. To this end, fix an $\varepsilon > 0$. It follows from (2) that there exists a

$t_\varepsilon > t_*$ such that

$\int_{t_*}^{\infty} h(s)ds < \frac{\varepsilon}{9(p_1(b)+ p_2(b))}$

Since the function f:$[t_*,t_\varepsilon]$ x $[-t_\varepsilon b, t_\varepsilon b]$ x $[-b,b] \to \mathbb{R}$ is uniformly continuous, by definition, there exists an $\eta_\varepsilon > 0$ such that

$|f(t,u_1,v_1) - f(t,u_2,v_2)| < \frac{\varepsilon}{9t_\varepsilon}$

for all $t \in [t_*,t_\varepsilon]$, all $u_1,u_2 \in [-t_\varepsilon b, t_\varepsilon b]$ satisfying $|u_1 - u_2| < \eta_\varepsilon$, and all $v_1, v_2 \in$
$[-b,b]$

A straightforward computation leads to the following estimates:

$|(Tu)'(t) - (Tv)'(t)| \le \int_{t_*}^{\infty} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))|\, ds$

$\le \int_{t_*}^{t_\varepsilon} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))|\, ds + \int_{t_\varepsilon}^{\infty} |f(s,u(s),u'(s))|\, ds$

$+ \int_{t_\varepsilon}^{\infty} |f(s,v(s),v'(s)))|\, ds$

$\le \frac{\varepsilon}{9t_\varepsilon}(t_\varepsilon - t_*) + 2(p_1(b) + p_2(b))\int_{t_\varepsilon}^{\infty} h(s)ds < \frac{\varepsilon}{3}$

and

$\frac{|(Tu)(t)-(Tv)(t)|}{t} \le \int_{t_*}^{\infty} \left|f\left(s,u(s),u'(s)\right) - f(s,v(s),v'(s))\right| ds$

$+\int_{t_*}^{t} \left|f\left(s,u(s),u'(s)\right) - f(s,v(s),v'(s))\right|\, ds$

$\le 2\int_{t_*}^{\infty} |f(s,u(s),u')) - f(s,v(s),v'(s))|\, ds < 2\frac{\varepsilon}{3},$

which are valid for all $t > t_*$ and all $u,v \in C$ satisfying $\|u - v\| < \eta_\varepsilon / t_\varepsilon$. Recalling the definition of the norm in the function space $V(t_0)$, we conclude that

$\|Tu - Tv\| \le \varepsilon$

For all $u,v \in C$ satisfying $\|u - v\| < \delta(\varepsilon) = \eta_\varepsilon / t_\varepsilon$. Therefore, the continuity of the operator T is established.

In order to show that TC is relatively compact, we have to prove that TC satisfies hypotheses $(i)_1$, $(i)_2$, $(ii)_1$, $(ii)_2$ and $(iii)_1$ of Proposition 3. First, we note that assumptions $(i)_1$ and $(i)_2$ follow from the fact that $TC \subset C$.

To verify hypotheses $(ii)_1$ and $(ii)_2$, for any $t_2 \ge t_1 \ge t_*$ and any $u \in C$, we obtain by a straightforward computation the following estimates:

$|(Tu)'(t_2) - (Tu)'(t_1)| \le \int_{t_1}^{t_2} |f(s,u(s),u'(s))|\, ds$

$\le [p_1(b) + p_2(b)] \int_{t_1}^{t_2} h(s)ds$

and

$\left|\frac{(Tu)(t_2)}{t_2} - \frac{(Tu)(t_1)}{t_1}\right| \le |u_0|\left(\frac{1}{t_1} - \frac{1}{t_2}\right) + |a|t_*\left(\frac{1}{t_1} - \frac{1}{t_2}\right)$

$+ t_*\left(\frac{1}{t_1} - \frac{1}{t_2}\right) \int_{t_*}^{\infty} |f(s,u(s),u'(s))|\, ds$

$+ \left|\int_{t_*}^{t_2} \frac{(t_2-s)}{t_2}\right| f\left(s,u(s),u'(s)\right) ds - \int_{t_*}^{t_1} \frac{(t_1-s)}{t_1} f(s,u(s),u'(s)ds|$

$\le [|u_0| + |a|t_* + t_*\left(p_1(b) + p_2(b)\right)H](t_2 - t_1)$

$+ \int_{t_1}^{t_2} |f(s,u(s),u'(s))|\, ds + \int_{t_1}^{t_2} \frac{s}{t_2} |f(s,u(s),u'(s))|\, ds$

$+ \int_{t_*}^{t_1} s\left(\frac{1}{t_1} - \frac{1}{t_2}\right) |f(s,u(s),u'(s))|\, ds$

$\leq [|u_0| + |a|t_* + t_*H(p_1(b) + p_2(b))](t_2 - t_1)$

$+2\int_{t_1}^{t_2}|f(s,u(s),u'(s))|\,ds + (t_2 - t_1)\int_{t_*}^{t_1}|f(s,u(s),u'(s))|\,ds$

$\leq [|u_0| + |a|t_* + H(1 + t_*)(p_1(b) + p_2(b))](t_2 - t_1)$

$+ 2(p_1(b) + p_2(b))\int_{t_1}^{t_2} h(s)ds.$

Assumptions $(ii)_1$ and $(ii)_2$ follow now from the above estimates and condition (2). In order to verify hypothesis $(iii)_1$ in Proposition 3, observe that

$$|(Tu)'(t) - a| \leq \int_t^\infty |f(s,u(s),u'(s))|\,ds$$

$$\leq [p_1(b) + p_2(b)]\int_t^\infty h(s)ds$$

for all $t \geq t_*$ and all $u \in C$ due to the fact that

$^aT_u = a + \lim_{t \to -\infty} \int_t^\infty f(s,u(s),u'(s))\,ds = a$

for any $u \in C$.

Thus, all the assumptions of Proposition 3 are verified and we conclude that the set TC is relatively compact. Consequently, one can now apply the Schauder-Tikhonov theorem, according to which the operator T has a fixed point $u(t)$ in C. This fixed point is a solution of the following bilocal problem

$u'' + f(t,u,u') = 0,\ t \geq t_*,$

$u(t_*) = u_0,$

$\lim_{t \to +\infty} u'(t) = a.$ (5)

Step 2: Denote by $u(t; u_0, a)$ the solution of problem (5) whose existence has been proved in Step 1. Consider now the Cauchy problem.

$u'' + f(t,u,u') = 0,\quad t \geq t_0,$

$u(t_*) = u_0,$

$u'(t_*) = u_1.$ (6)

where u_1 stands for $u'(t_*;u_0,a)$. We shall show that problem (6) has a solution in $[t_o,t_*]$. Consider the operator $T:C^1([t_0,t_*],\mathbb{R}) \to C^1([t_0,t_*],\mathbb{R})$ defined by the following formula:

$(Tu)(t) = u_0 - u_1(t_* - t) - \int_t^{t_*}(s - t)f(s,u(s),u'(s))\,ds,\ t \in [t_0,t_*].$

Applying Lemma 3 and using the argument similar to that exploited in Step 1, it is not difficult to deduce that for every bounded subset M of $C^1([t_0,t_*],\mathbb{R})$, the operator $T:M \to C^1([t_o,t_*],\mathbb{R})$ is uniformly continuous and the set TM is relatively compact. To simplify the computation, instead of using inequality (1), one can use the less restrictive inequality.

$|f(t,u,v)| \leq h(t)\left[p_1\left(\frac{|u|}{t}\right) + p_2(|v|)\right]$

$\leq h(t)[p_1(|u|) + p_2(|v|)].$

According to Lemma (1), in order to prove that the operator T has a fixed point in $C^1([t_0,t_*],\mathbb{R})$ one has to show that the set

$E(T) = \{\chi \in C^1([t_0,t_*],\mathbb{R}): \chi = \lambda T\chi,\ 0 < \lambda < 1\}$

is bounded. To this end, pick a function $u \in E(T)$. Then there exists a $\lambda \in (0,1)$ such that

$u(t) = \lambda u_0 - \lambda u_1(t_* - t) - \lambda \int_t^{t_*}(s-t)f(s,u(s),u'(s))\,ds$

for all $t \in [t_0,t_*]$. It follows from the latter equality that

$$|u(t)| \leq |u_0| + |u_1|t_* + \int_t^{t_*} sh(s)[p_1(|u(s)|) + p_2(|u'(s)|)]\,ds \qquad (7)$$

and

$$|u'(t)| \leq |u_1| + \int_t^{t_*} h(s)[p_1(|u(s)|) + p_2(|u'(s)|)]\,ds \qquad (8)$$

for all $t \in [t_0,t_*]$.

Introducing for $s \geq 0$ the following notation

$K = t_0 + |u_0| + (1+t_*)|u_1|, \quad z(t) = |u(t)| + |u'(t)|.$

$g(s) = p_1(s) + p_2(s),$

we can write inequalities (7) and (8) in the form

$$z(t) \leq K + \int_t^{t_*} sh(s)g(z(s))ds, \; t \in [t_0 t_*].$$

Applying now Lemma 2, part (i), we conclude that

$z(t) \leq G^{-1}(G(K) + t^*H) = K_* < +\infty$

for all $t \in [t_0, t_*]$. This implies that $\|u\| \leq K_*$ for all $u \in E(T)$. Thus, we have proved that the set $E(T)$ is bounded. Therefore, by the Schaefer theorem (Lemma 1) we conclude that the operator T has a fixed point $u(t)$ in $C^1([t_0, t_*], \mathbb{R})$ and this fixed point is a solution of the Cauchy problem (6).

Step 3: Denote by $v(t;u_0,u_1)$ the solution of problem (6) whose existence has been proved in Step 2. Then the function $u(t)$ defined by the formula

$$u(t) = \begin{cases} v(t;u_0,u_1), & t \in [t_0,t_*]. \\ u(t;u_0,a), & t \geq t_* \end{cases}$$

is the desired solution of the problem (3). The proof of Theorem (1) is complete □

Theorem 2: Suppose that there exist continuous functions h, p_1, p_2 : $\mathbb{R}_+ \to \mathbb{R}_+$ such that the following inequality holds:

$$|f(t,u,v)| \leq h(t)\left[p_{1\left(\frac{|u|}{t}\right)} + p_2(|v|)\right], \qquad \text{(i)}$$

where the functions p1(s),p2(s) are positive, non-decreasing, and the function h(s) satisfies

$$\int_{t_0}^{\infty} h(s)ds = H < +\infty \qquad \text{(ii)}$$

Furthermore, assume that $G(+\infty) = +\infty$, where for $x \geq t_0$

$$G(x) = \int_{t_0}^{x} \frac{(ds)}{p_{1(s)} + p_{2(s)}}$$

Then, for every $u_0, u_1 \in \mathbb{R}$, the initial value problem

$u'' + f(t,u,u') = 0 \qquad t \geq t_0$

$u(t_0) = u_0,$

$u'(t_o) = u_1$ (iii)

has at least one solution u(t) defined on $[t_{0,+\infty})$ with the asymptotic representation at+b as $t\to\infty$.

Proof: Define the operator T**:** $V(t_0) \to V(t_0)$ the formula

$$(T_u)(t) = u_0 + u_1(t-t_0) - \int_{t_o}^{t} (t-s) f(s, u(s), u'(s)) ds,\ t \geq t_0 \qquad (9)$$

Acting as in the proof of theorem (1), one can show that the operator T is completely continuous. Then, it is necessary to verify the hypothesis of proposition 3 demonstrating that the set TC is relatively compact. A straightforward computation combined with the application of Lemma 2, part (i) leads to the conclusion that the set $E(T)$ is bounded. This in turn implies that by the Schaefer theorem (Lemma 1), the operator T has a fixed point $u(t)$ in $V(t_0)$ that corresponds to the solution of Eq. (1) with the desired asymptotic behaviour. □

Theorem 3: Suppose that conditions (1) and $G(+\infty) = +\infty$ hold while (2) is replaced with a stronger assumption

$$\int_{t_0}^{\infty} sh(s) ds = H_* < +\infty$$

Then for every pair u_0, u_1 of real numbers, the initial value problem (iii) has at least one solution $u(t)$ defined on $[t_0, +\infty)$ with the asymptotic representation $u(t) = at + b + o(1)$ as $t \to +\infty$, where a,b are real constants.

Proof: The pattern of the proof repeats that of theorem 1, but the operator defined by (9) now acts in $W(t_0)$. This requires application of Proposition 4 for the verification of the fact that the set TC is relatively compact in $W(t_0)$.

Finally, the following result is dual to theorem 3 and establishes for any given pair of real numbers a and b, the existence of solution $u(t)$ of Eq. (1) with the asymptotic representation $u(t) = at + b + 0(1)$ at infinity.

Theorem 4: Suppose that the hypotheses of theorem 3 are satisfied. Then for every a, b $\in \mathbb{R}$, there exist a solution $u(t)$ of Eq. (1) defined on $[t_0, +\infty)$ which has the asymptotic representation $u(t) = at + b + o(1)$ as $t \to \infty$.

Proof: Define the operator $T: W(t_0) \to W(t_0)$ by the formula

$$(T_u)(t) = at + b + \int_{t}^{\infty} (t-s) f(s, u(s), u'(s))\, ds,\ t \geq t_0$$

Following the same lines as in the proof of theorem 1, one can show that the operator T is completely continuous. Computation similar to that carried out in the proof of theorem 1 and successful application of Lemma 2, part (ii) yields the fact that the set $E(T)$ is bounded and Schaefer theorem (Lemma 1) is applied, to conclude that the operator T has a fixed point $u(t)$ in $W(t_0)$ that corresponds to the solution of Eq. (1).

5. *Discussion and Conclusion*

The case $G(+\infty) < +\infty$

The non-linear differential equation

$$u'' - \frac{2}{t^4} u^2 = o, \qquad t \geq 1 \qquad (10)$$

has been considered by Meng (2) as a counter example to the result reported by Tong [6, Theorem 2]. One has

$$h(t) = \frac{2}{t^2}\,,\ p_1(s) = s^2 + \frac{1}{2},\quad p_2(s) = \frac{1}{2}$$

and it is easy to check that all the hypotheses of theorem 2 are satisfied for Eq. (10) except for the crucial assumption.

$$G(+\infty) = +\infty \qquad (11)$$

As a result of violation of condition (11), not all solutions of Eq. (10) exhibit the desired "linear-like" asymptotic behaviour at infinity and such solution is $u(t) = t^2$.

This leads to the conclusion that if condition (11) fails to hold, various inconsistencies regarding global existence or prescribed behaviour of solutions of Eq.(1) at infinity might occur.

The benefit of this paper is the derivation of the sufficient conditions for existence of global solutions and investigation of a type of asymptotic behavior of solutions of second-order nonlinear differential equations.

References

1. C. Avramescu. Sur l'existence des solutions convergentes de systemes d''equations differentielles nonlineaires, Ann. Mat. Pura Appl. 81 (4) 147-168.
2. R. Bellman. Stability Theory of Differential Equations, Mc Graw-Hill, London, 1953.
3. E. Hille. Non oscillation Theorems, Trans. Amer. Math. Soc. 64(1948). 234-252.
4. P. Hartman, A. Wintner. On non-oscillatory linear Differential Equations, Amer. J. Math. 75(1953) 717-730.
5. A.G. Kartsatos. Advanced Ordinary Differential Equations, Mariner Publishing Company, Inc., Tampa, FL, 1980.
6. J.P. Lasalle, S. Lefschetz. Stability by Liapunov's Direct Method with Applications, Academic Press, New York, 1961.
7. R. Moore, Z. Nehari. Non-oscillation Theorems for a class of Nonlinear Differential Equations, Trans. Amer. Math. Soc. 93(1959) 30-52.
8. F.W. Meng. A Note on Tong Paper: The asymptotic behaviour of a class of Nonlinear Differential Equations of Second Order, Proc. Amer. Math. Soc. 108(1990) 383-386.
9. Z. Nehari. On a class of Nonlinear Second-Order Differential Equations, Trans. Amer. Math. Soc. 95(1960). 101-123.
10. G. Octavian Mustafa, V. Yuri Yogovchenko. Global Existence of Solutions with Prescribed Asymptotic Behavior of Second-Order Nonlinear Differential Equations. Nonlinear Analysis 51 (2002). 339-368.
11. P. Souplet. Existence of exceptional growing-up Solutions for a Class of Nonlinear Second-Order Ordinary Differential Equations, Asymptotic Anal. 11(1995) 185-207.
12. J. Tong. The asymptotic behavior of a class of nonlinear differential equations of second order, Proc. Amer. Math. Soc. 14(1963). 12-14.
13. J.S.W. Wong. On Second Order Nonlinear Oscillation, Funkcial. Ekvac. 11(1968) 207-234.
14. P. Waltman. On the Asymptotic Behavior of Solutions of a Nonlinear Equation, Proc. Amer. Math. Soc. 15 (1964) 918-923.