
SOME CHARACTERIZATION OF OSCULATING CURVES ACCORDING TO DARBOUX FRAME IN THREE DIMENSIONAL EUCLIDEAN SPACE

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Abstract

In this paper, we give some characterizations of osculating curves according to Darboux frame in 3 –dimensional space, we define osculating curve in E^3 as a curve whose position vector always lies in their osculating plane, we also classify such curves in terms of their curvature functions, and we give some examples.

Keywords: Osculating Curves; Darboux Frame; Euclidean Space; Curvature

1 Introduction

In differential geometry of curves, it is well known that at every point of a regular curve α , planes that spanned through $\{T, N\}$, $\{N, B\}$, $\{B, T\}$ are alluded to as osculating plane, Normal plane and Rectifying plane respectively and the curves lying in them are called Osculating curves, normal curves, and rectifying curves respectively. The osculating curves and rectifying curves in $\mathbf{E}^3$ are studied by [5, 6]. In [5], the authors gave osculating curves some characterizations in $\mathbf{E}^3$, then they gave the definition of osculating osculating in $\mathbf{E}^4$ as the curve whose position vector always lies in their osculating plane, which satisfies the equation $\alpha(s) = \lambda T + \mu N + VB_2$, where $\lambda, \mu, V \in C^\infty$.

In $\mathbf{E}^3$ space, the Darboux frame is one of the most familiar frame fields along a space, it was constructed from the curve's velocity and the normal surface vector, where the velocity and the acceleration of the curve were configured as the Frenet-serret frame. The derivatives of these frames, provide some real valued functions, termed as the curvature and torsion of the Frenet-serret frame, and the normal curvature, geodesic curvature and torsion for Darboux frame [3, 4, 11].

In this work, for a given regular curves which are lying fully in osculating plane in $\mathbf{E}^3$, we study some special osculating curves with reference to Darboux frame in the three dimensional Euclidean space $\mathbf{E}^3$. We look up to seeing our study results aid in further studies so much for researchers in mathematics and other disciplines.

2 Preliminaries

Let $\alpha: \mathbb{R} \rightarrow \mathbf{E}^3$ be an arbitrary Curve in $\mathbf{E}^3$. α is indicated to be a unit speed or in other word parameterized by arc-length curve s if $\langle \alpha'(s), \alpha'(s) \rangle = 1$. Also, the measure inner product of $\mathbf{E}^3$ is;

$$\langle P, Q \rangle = p_1 q_1 + p_2 q_2 + \dots + p_n q_n.$$

for each $P = (p_1, p_2, \dots, p_n)$, $Q = (q_1, q_2, \dots, q_n) \in \mathbb{R}^n$. Norm of the vector $P \in \mathbf{E}^3$ is given by; $\|P\| = \sqrt{\langle P, P \rangle}$.

A moving frame of a curve with arc-length parameter s in euclidean space $\mathbf{E}^3$, it's described in even more detail as a 4 -type of vectors drawn from $\mathbf{E}^3, (\alpha(s), T, L, M)$, where $\alpha(s)$ is a choice on α and (T, L, M) is an orthonormal basis of the vector space $\mathbf{E}^3$ based on $\alpha(t)$. A moving frame in mathematics is a versatile development for the notion of an ordered basis on vector spaces that has often been used to study the extrinsic differential geometry of smooth manifolds embedded in a homogeneous space. Since the frame (T, L, M) is orthonormal under the inner product of euclidean space $\mathbf{E}^3$, the following matrix gives the change of the frame and its derivative:

$$\begin{bmatrix} T \\ L \\ M \end{bmatrix}' = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ L \\ M \end{bmatrix} \quad (1)$$

where the exterior derivative $T(s)$ with respect to s decomposes uniquely, while k_3 measures the twisting of this framing. The frenet-serret frame, Bishop frame and Darboux frame on a curve are some simple examples of a moving frame. Now lets see these facts using the frame given in Eq. (1).

Assume that the curvature vectoreof the curve α never vanishes then we can write that k_2 is zero and $k_1 = \|T'(s)\|$. In this case $L = N$ is a "normal vector" of the frenet

frame $\{T, N, B = T \times N\}$, whose turning away of the plane of curvature k_3 is the torsion which determines how firmly it turns out from the curvature of the plane. If Eq. (1) is defined by the condition that $k_3 = 0$ then, we can easily obtain Bishop frame. Bishop in [5] defined a new frame curve and named it Bishop frame that is well defined even though the second derivative in the curve has vanished in E^3 . Using $L = N_1$ and $M = N_2$ we get the alternative moving frame defined in [7].

Let the curves $\psi: I \subset R \rightarrow E^3$ be an arbitrary Curve in E^3 where the position vector ψ that is always in the rectifying plane are just called rectifying curves. Similarly, normal curves and osculating curves position vectors all lies in their respective planes. A normal curve, rectifying curve and osculating curve their position vector ψ by definition adopts the following equation, accordingly;

$$\begin{aligned}\psi(s) &= a_1(s)N(s) + a_2(s)B(s) \\ \psi(s) &= b_1(s)T(s) + b_2(s)B(s) \\ \psi(s) &= c_1(s)T(s) + c_2(s)N(s)\end{aligned}\quad (2)$$

for differentiable functions $a_i, b_i, c_i s \in I \subset R, i = 1, 2..$ [5].

2.1 Darboux Frame in E^3

Definition 2.1 Let $\zeta \subset E^3$ be a surface and oriented, $\alpha: R \rightarrow \zeta$ be a parameterized curve, for a given unit tangent vector field $T(s)$ of α and a unit normal vector field $M(s)$ of ζ subjected to the curve α , then the frame $\{T, L, M\}$ is said to be Darboux frame along the curve α , where $L = M \times T$. The derivative formula of the Darboux frame $\alpha(s)$ is given in the following form:

$$\begin{bmatrix} T \\ L \\ M \end{bmatrix}' = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ L \\ M \end{bmatrix}\quad (3)$$

where k_n, τ_g and k_g denote the normal curvature, geodesic torsion and geodesic curvature of the curve α respectively [12].

Definition 2.2 Let ζ be a surface in E^3 and α is an arbitrary curve of a Darboux frame $\{T, L, M\}$ on the surface ζ , the following functions are defined by;

$$\begin{aligned}k_g(t) &= \frac{1}{\|\alpha'(t)\|^2} \langle \alpha''(t), L(t) \rangle \\ k_n(t) &= \frac{1}{\|\alpha'(t)\|^2} \langle \alpha''(t), M(t) \rangle \\ \tau_g(t) &= \frac{1}{\|\alpha'(t)\|} \langle L'(t), M(t) \rangle\end{aligned}$$

in [8].

Definition 2.3 We can say that in [12] α is;

$$\text{Geodesic curve} \Leftrightarrow k_g \equiv 0$$

$$\text{Asymptotic curve} \Leftrightarrow k_n \equiv 0$$

$$\text{Line of curvature} \Leftrightarrow \tau_g \equiv 0$$

Also, we can say α is:

$$\underline{\kappa, \tau \quad \alpha}$$

$\kappa \equiv 0$ if and only if	a straight line.
$\tau \equiv 0$ if and only if	a plane curve.
$\kappa \equiv \text{constant} > 0, \tau \equiv \text{constant} > 0$ if and only if	a W – curve or circular helix.
$\frac{\tau}{\kappa} \equiv \text{constant}$ if and only if	a generalize helix.
$\kappa \equiv \text{constant}, \tau \neq \text{constant}$ if and only if	Salkowski curve [10,13].
$\kappa \neq \text{cons.}, \tau \equiv \text{cons.}$ if and only if	Anti – salkowski curve [13].

2.2 Osculating Curve According To Darboux Frame in E^3

In [2], Chen implemented the osculating curves as a space curve for which the curve's position vector spanned by tangent and normal vector field are lying always in their osculating planes.

Definition 2.4 let α then be an osculating curve, thus

$$\alpha(s) = C_1 T(s) + C_2 L(s) \quad (4)$$

[label=)]

1. If $C_2(s) = C = \text{constant}$ is taken; $\alpha(s) = C_1(s)T(s) + CL(s)$ is define as a type 1 osculating-type curve.
2. If $C_1(s) = C = \text{constant}$ is taken, then $\alpha(s) = C(s)T(s) + C_2(s)L(s)$ is define as type 2 osculating-type curve [5].

Theorem 2.5 let $\alpha(s)$ be a unit speed regular curve entirely in E^3 . As such, α is an osculating curve if and only if α is a planar curve or straight line. [5].

Theorem 2.6 For a given parameterized osculating curve in E^3 , then the tangential component $C_1(\langle \alpha(s), T(s) \rangle = C_1)$ and the normal component $C_2, (\langle \alpha(s), C_2(s) \rangle = C_2)$ the curve's position vector which appeases below;

$$C_1^2 + C_2^2 = 2 \int C_1 ds \quad (5)$$

Conversely, if the tangential and principal normal component $(\langle C_1(s), C_2(s) \rangle = C_1)$ of curve's position vector $\alpha(s)$ in E^3 satisfies Eq.(5) therefore $\alpha(s)$ is an osculating or rectifying curve [5].

3 Some Characterization of Osculating Curves According to Darboux Frame in E^3

In this section we study the characterization of osculating type curves in E^3 with respect of their curvatures.

Theorem 3.1 Suppose $\alpha: I \subset R \rightarrow \zeta$ is a parameterized curve in E^3 , with non-zero curvatures k_n, k_g and τ_g . Then, α is congruent to an osculating curve provided that;

$$\left[\frac{1}{k_g} \left(e^{\int k_g \frac{\tau_g}{k_n} ds} \right)' \right]' + k_g e^{\int k_g \frac{\tau_g}{k_n} ds} = -1 \quad (6)$$

Proof:

Where by fact, a position vector of a given curve α is certained by Def. (2.4), for some functions $C_1(s)$ and $C_2(s)$ which are differentiable. By using (4) and (3).

$$\alpha(s) = C_1(s)T(s) + C_2(s)L(s)$$

$$T = (C_1' - C_2 k_g)T + (C_1 k_g + C_2')L + (C_1 k_n + C_2 \tau_g)M \quad (7)$$

it follows that;

$$\left. \begin{aligned} C_1' - C_2 k_g &= 1 \\ C_1 k_g + C_2' &= 0 \\ C_1 k_n + C_2 \tau_g &= 0 \end{aligned} \right\} \quad (8)$$

by using the relations in (8) we then obtained the following results;

$$\frac{\tau_g}{k_n} = -\frac{C_1}{C_2} \quad (9)$$

$$C_1 = -\frac{C_2'}{k_g}. \quad (10)$$

By substituting (10) in (9) we get;

$$\frac{C_2'}{C_2} = -\frac{k_g}{k_n} \tau_g$$

by integrating the above equation, we obtain;

$$C_2 = e^{\int k_g \frac{\tau_g}{k_n} ds}. \quad (11)$$

$C_1(s)$ is obtained by substituting (11) in (10)

$$C_1 = -\frac{1}{k_g} \left(e^{\int k_g \frac{\tau_g}{k_n} ds} \right)', \quad (12)$$

by using the first equation in (8), (11) and (12), then the relation in (6) holds.

Theorem 3.2 Suppose $\alpha: I \subset R \rightarrow \zeta$ is parameterized curve in E^3 , with k_g , τ_g and $k_n \neq 0$ as geodesic curvature, geodesic torsion and normal curvature respectively. Then α is congruent to type 1 – osculating curve if and only if:

$$\left(\frac{\tau_g}{k_n} \right)' + k_g = -\frac{1}{C}, C \in R_0 \quad (13)$$

proof: From (2.4), the type 1 – osculating curve α is given by:

$$\alpha(s) = C_1(s)T(s) + CL(s) \quad (14)$$

where $C_1(s)$ is a differentiable function and C a constant function. Differentiating Eq. (14) and using the Darboux frame Eq. (3).

$$\alpha(s) = C_1(s)T(s) + CL(s)$$

$$T = (C_1' - C k_g)T + C_1 k_g L + (C_1 k_n + C \tau_g)M \quad (15)$$

it follows that:

$$\left. \begin{aligned} C_1' - C k_g &= 1 \\ C_1 k_g &= 0 \\ C_1 k_n + C \tau_g &= 0 \end{aligned} \right\} \quad (16)$$

from (16) above

$$C_1 = -C \frac{\tau_g}{k_n}. \quad (17)$$

By substituting (17) in the first equation of (16) we get,

$$\left(\frac{\tau_g}{k_n} \right)' + k_g = -\frac{1}{C}$$

which satisfies (13). conversely, the curvatures k_g , k_n and τ_g are presumed to be an arbitrary parameterized curve $\alpha(s) \in E^3$, which satisfies (13), let's refer the vector $V \in E^3$ indicated by;

$$V(s) = \alpha(s) + C \frac{\tau_g}{k_n} T(s) - CL(s) \quad (18)$$

by using the relation in (3), (13) and differentiating (18) we obtain the following result;

$$V'(s) = \left(1 + C \left(\frac{\tau_g}{k_n}\right)' + C k_g\right) T(s) + C \frac{\tau_g}{k_n} k_g L(s) + \left(C \frac{\tau_g}{k_n} k_n - C \tau_g\right) M(s)$$

$$V'(s) = 0$$

implies V is a constant vector, which shows $\alpha(s)$ is congruent to an osculating curve. Any arbitrary curve that has constant curvature functions in E^3 is said to be a W –curve [9]. Such curves satisfy theorem (3.2).

The subsequent theorems describe the characterization of a W –curve in E^3 as for osculating curves:

Theorem 3.3 Every parameterized W –curve with non-zero k_g , k_n and τ_g in E^3 , is congruent to a type 1 –osculating curve.

The proof is obvious from theorem (3.2).

Theorem 3.4 None of a parameterized W –curve with k_g , k_n and τ_g curvatures in E^3 is a type 2 – osculating curve.

Proof: From (2.4), the type 2 – osculating curve is given by:

$$\alpha(s) = CT(s) + C_2(s)L(s). \quad (19)$$

where $C_2(s)$ is a differentiable function and C is a constant function. By using (17) and (3) we obtain;

$$T(s) = -C_2 k_g T(s) + (C k_g + C_2'(s))L(s) + (C k_n + C_2(s)\tau_g)M(s) \quad (20)$$

it follows that;

$$\left. \begin{aligned} -C_2 k_g &= 1 \\ C_2 &= -\frac{1}{k_g} \\ C k_g + C_2' &= 0 \\ C k_n + C_2 \tau_g &= 0 \end{aligned} \right\} \quad (21)$$

By taking the second and third equations of (21) we get;

$$C k_g + \frac{k_g'}{k_g^2} = 0 \quad (22)$$

Theorem 3.5 Suppose $\alpha: R \rightarrow \zeta$ is a parameterized curve in E^3 with non-zero k_n and τ_g , either there exist a geodesic curve which is congruent to a type 2 – osculating curve or its geodesic curvature is given by;

$$k_g = \sqrt{\frac{1}{2(sC - C_3)}} \quad (23)$$

The proof can be seen clearly from theorem (3.4).

4 Application

Throughout this section we try to provide examples of applications to our work.

Example 4.1 Consider a given oriented surface $\zeta \subset E^3$

$$\zeta(u, v) = \alpha(u) + (\sqrt{3}\cos(u), \sqrt{3}\sin(u), v)$$

and $\alpha: I \subset R \rightarrow \zeta$ be an arbitrary curve such that (Figure.1)

$$\alpha(u) = (\sqrt{3}\cos(u), \sqrt{3}\sin(u), u)$$

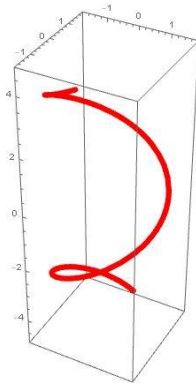


Figure 1: is a 3D arbitrary curve.

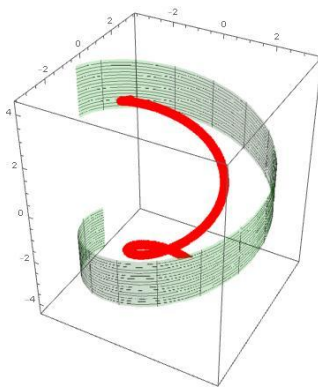


Figure 2: is the 3D arbitrary curve on the surface ζ

$$T = \frac{\alpha'(u)}{\|\alpha'(u)\|} = \left(-\sqrt{\frac{3}{7}}\sin(u), \sqrt{\frac{3}{7}}\cos(u), \frac{1}{\sqrt{7}} \right),$$

T is the tangential vector field of the position vector $\alpha(u)$. The normal vector field M of the surface ζ can be calculated as follows;

$$M = \frac{\zeta_u \times \zeta_v}{\|\zeta_u \times \zeta_v\|} \\ = (\cos(u), \sin(u), 0)$$

from Definition(2.1),

$$L = M \times T \\ = \left(\frac{1}{\sqrt{7}}\sin(u), -\frac{1}{\sqrt{7}}\cos(u), \sqrt{\frac{3}{7}} \right).$$

According to Definition (2.2) k_g , k_n and τ_g of the curve $\alpha(u)$ are obtained as;

$$k_g = \frac{1}{\|\alpha'(u)\|^2} \langle \alpha''(u), L \rangle \\ k_g = 0.$$

now, we need to find k_n .

$$k_n = \frac{1}{\|\alpha'(u)\|^2} \langle \alpha''(u), M \rangle$$

$$= \frac{-\sqrt{3}}{7}.$$

similarly, we find τ_g .

$$\tau_g = \frac{1}{\|\alpha'(u)\|} \langle L', M \rangle$$

L' gotten as;

$$L' = \left(\frac{1}{\sqrt{7}} \cos(u), \frac{1}{\sqrt{7}} \sin(u), 0 \right)$$

$$\tau_g = \frac{1}{\sqrt{7}} \left\langle \left(\frac{1}{\sqrt{7}} \cos(u), \frac{1}{\sqrt{7}} \sin(u), 0 \right), (\cos(u), \sin(u), 0) \right\rangle$$

$$= \frac{1}{7}.$$

clearly, α is congruent to a type-1 osculating curve which satisfies theorem (3.2) and (3.5). Similarly, it is a W curve from theorem (3.3) and (3.4).

Example 4.2 Let us consider $\zeta \subset E^3$ as a given oriented parameterized surface

$$\zeta(s, t) = \alpha(s) + t(1, 1, 0)$$

with a base curve in the form below;

$$\alpha(s) = \left(\frac{(1+s)^3}{3}, \frac{(1-s)^3}{3}, \frac{s}{\sqrt{2}} \right)$$

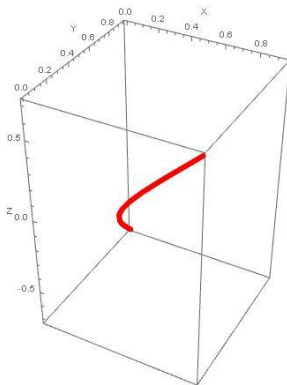


Figure 3: is a 3D unit speed curve.

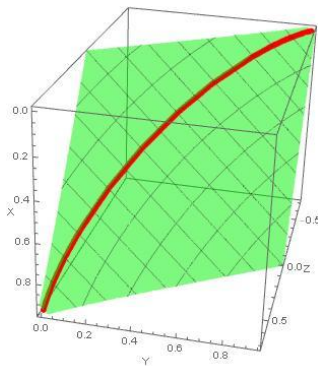


Figure 4: is the 3D unit speed curve on the surface ζ

the Darboux frame can obtain as follows;

$$T = \alpha'(s) = \left(\frac{(1+s)^2}{2}, -\frac{(1-s)^2}{2}, \frac{1}{\sqrt{2}} \right),$$

$$\zeta_t(s, t) = (1, 1, 0)$$

$$\begin{aligned} M &= \frac{\zeta_s \times \zeta_t}{\|\zeta_s \times \zeta_t\|} \\ &= \frac{\sqrt{2}}{(3+\sqrt{1-s^2})^{\frac{1}{2}}} \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{(1+s)^{\frac{1}{2}}}{2} + \frac{(1-s)^{\frac{1}{2}}}{2} \right) \\ &= \left(-\frac{1}{(3+\sqrt{1-s^2})^{\frac{1}{2}}}, \frac{1}{(3+\sqrt{1-s^2})^{\frac{1}{2}}}, \frac{\sqrt{2}((1+s)^{\frac{1}{2}} + (1-s)^{\frac{1}{2}})}{2(3+\sqrt{1-s^2})^{\frac{1}{2}}} \right). \end{aligned}$$

$$\begin{aligned} L &= M \times T \\ &= \left(\frac{\sqrt{2}(3-s) + \sqrt{2}(1-s)^{\frac{1}{2}}}{4(3+\sqrt{1-s^2})^{\frac{1}{2}}}, \frac{\sqrt{2}(3+s) + \sqrt{2}(1-s)^{\frac{1}{2}}}{4(3+\sqrt{1-s^2})^{\frac{1}{2}}}, \frac{(1-s)^{\frac{1}{2}} - (1+s)^{\frac{1}{2}}}{2(3+\sqrt{1-s^2})^{\frac{1}{2}}} \right) \end{aligned}$$

$$k_g = \frac{1}{\|\alpha'(s)\|^2} \langle \alpha''(s), L \rangle$$

$$\begin{aligned} k_g &= \frac{\sqrt{2}}{16(3+\sqrt{1-s^2})^{\frac{1}{2}}} \left((1+s)^{-\frac{1}{2}}(3-s) + (1+s)^{-\frac{1}{2}}(1-s^2)^{\frac{1}{2}} \right. \\ &\quad \left. + (1-s)^{-\frac{1}{2}}(3+s) + (1-s)^{-\frac{1}{2}}(1-s^2)^{\frac{1}{2}} \right) \end{aligned}$$

$$\begin{aligned} k_n &= \frac{1}{\|\alpha'(s)\|^2} \langle \alpha''(s), M \rangle \\ &= \frac{(1-s)^{-\frac{1}{2}} + 2(1+s)^{-\frac{1}{2}}}{4(3+\sqrt{1-s^2})^{\frac{1}{2}}} \end{aligned}$$

L' is found as,

$$\begin{aligned} L' &= \left(\frac{-\sqrt{2} - \frac{s\sqrt{2}}{\sqrt{1-s^2}}}{4\sqrt{3+\sqrt{1-s^2}}} + \frac{s((3-s)\sqrt{2} + \sqrt{2}\sqrt{1-s^2})}{8\sqrt{1-s^2}(3+\sqrt{1-s^2})^{\frac{3}{2}}}, \right. \\ &\quad \left. \frac{\sqrt{2} - \frac{s\sqrt{2}}{\sqrt{1-s^2}}}{4\sqrt{3+\sqrt{1-s^2}}} + \frac{s((3+s)\sqrt{2} + \sqrt{2}\sqrt{1-s^2})}{8\sqrt{1-s^2}(3+\sqrt{1-s^2})^{\frac{3}{2}}}, \right. \\ &\quad \left. \frac{(\sqrt{1-s} - \sqrt{1+s})}{4\sqrt{1-s^2}(3+\sqrt{1-s^2})^{\frac{3}{2}}} + \frac{\frac{-1}{2\sqrt{1-s}} - \frac{-1}{2\sqrt{1+s}}}{2\sqrt{3+\sqrt{1-s^2}}} \right) \end{aligned}$$

similarly, τ_g is found by;

$$\begin{aligned} \tau_g &= \frac{1}{\|\alpha'(s)\|} \langle L', M \rangle \\ &= \frac{\sqrt{2}(\sqrt{1-s^2} - 1)}{4\sqrt{1-s^2}(3+\sqrt{1-s^2})}. \end{aligned}$$

By taking $s = 0$, we get $k_g = \frac{\sqrt{2}}{8}$, $k_n = \frac{3}{16}$, $\tau_g = 0$. When we substitute the results, we find $C = -4\sqrt{2}$. Which clearly satisfies theorems 3.2 and from theorem (3.3) it is congruent to type-1 osculating curve.

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