
ON INTEGRABILITY OF SILVER RIEMANNIAN STRUCTURE

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Abstract

Exploring the geometry of Riemannian manifolds equipped with a silver structure is the focal focus of this study. A Φ – operator that is used on pure tensor fields is systematically applied for the study of integrability issue for Silver Riemannian structure. Nevertheless, the properties of twin Silver Riemannian metric and curvature properties for the silver Riemannian metrics are investigated ethically.

Keywords: Silver Structure; Riemannian Manifold; Pure tensor; Almost product structure; Twin metric.

1 Introduction

Suppose M is an infinitely differentiable manifold of n –dimension, the infinitely differentiable tensor field of type (s, t) is defined by the module over $\phi(M)$ on M . $\phi(M) \in C^\infty M$ functions of t covariant and s contravariant degree. It is assumed that Manifolds, connections and tensors are smooth.

The $(1,1)$ tensor field of constant rank called f – structure on a manifold, this concept was developed by Yano [1] satisfying the equation

$$f^3 + f = 0$$

The notion generalizes the almost contact and almost complex structures. In addition, Goldberg and yano [2] generalized it, by extending the degree to l of the polynomial structure a $(1,1)$ tensor field f of constant rank on M that satisfied the equation

$$\Theta(f) = f^l + af^{l-1} + \dots + a_2f + a_1I = 0$$

where I is of type $(1,1)$ identity tensor and $a_1, a_2, \dots, al$ are real numbers.

In [3], Hretcanu's golden structure on M was described by type $(1,1)$ tensor field J on M which satisfy the polynomial

$$J^2 = J + I$$

Then in [4], the golden structure on M is studied. However, the golden structure has been recently studied in [3, 5-14].

The positive solution of the equation $y^2 - 2y - 1 = 0$ gives another irrational number $\vartheta = 1 + \sqrt{2}$ which is as interesting as golden structure named as silver mean, silver number or silver ration. In [15], the silver number is used to define fractal geometry by Chandra and Rani, also ozkan [10] studied silver structure as a new structure on manifold.

Pure metrics in relation to other structures were studied by various authors in [1,5,16-36].

Silver Riemannian manifold is considerable because pure Riemannian metrics has to do with the silver structure.

Since Silver Riemannian structure and an almost product structure are related to each other, the method of Φ – operator used in theory of almost product structure can be applied to silver structure. Then, in this paper we try to study the polynomial structure $\vartheta^2 - 2\vartheta - I = 0$ by using its almost product structure $\vartheta^2 - I = 0$.

2 Preliminaries

Definition 2.1 Let M be a C^∞ differentiable manifold. A tensor of field ϑ of type $(1,1)$ on M is called a Silver structure on M if ϑ satisfies the equation;

$$\vartheta^2 = 2\vartheta + I \quad (1)$$

where I is the identity of $(1,1)$ tensor field on M [10].

Theorem 2.2 Provided that ϑ is a Silver structure on a manifold M , then

$$P = \frac{1}{\sqrt{2}}(\vartheta - I) \quad (2)$$

is said to be an almost product structure on M . conversely, any almost product structure P on M gives a silver structure on M as given below;

$$\vartheta = I + \sqrt{2}P \quad (3)$$

[10].

Suppose P is an almost product structure defined on a manifold M and g be a Riemannian metric such that;

$$g(P(X), P(Y)) = g(X, Y) \forall X, Y \in \chi M. \quad (4)$$

or similarly, P be a g –symetric endomorphism such as;

$$g(P(X), Y) = g(X, P(Y)) \forall X, Y \in \chi M. \quad (5)$$

the pair (g, P) is a Riemannian almost product structure [16, 20-30].

Definition 2.3 Suppose g is a Riemannian metric defined on a manifold M such that;

$$g(\vartheta(X), Y) = g(X, \vartheta(Y)) \forall X, Y \in \chi M \quad (6)$$

equation (6) can be written as;

$g(\vartheta(X), \vartheta(Y)) = g(\vartheta^2 X, Y) = ((2\vartheta + I)X, Y) = 2g(\vartheta X, Y) + g(X, Y) \forall X, Y \in \chi M$
Then, the pair (g, ϑ) is said to be a silver Riemannian structure and the triple (M, g, ϑ) is a silver Riemannian manifold [5]. Such Riemannian metrics are said to be pure metrics [5,22,25].

Definition 2.4 Suppose ϑ is a $(1,1)$ tensor field defined on M , a tensor field k of type (r, s) is called a pure tensor field with respect to ϑ if;

$$\begin{aligned} k(\vartheta X_1, X_2, \dots, X_s; Y^1, Y^2, \dots, Y^r) &= k(X_1, \vartheta X_2, \dots, X_s; Y^1, Y^2, \dots, Y^r) \\ &\vdots \\ &= k(X_1, X_2, \dots, \vartheta X_s; Y^1, Y^2, \dots, Y^r) \\ &\vdots \\ &= k(X_1, X_2, \dots, X_s; Y^1, Y^2, \dots, \vartheta Y^r) \end{aligned} \quad (7)$$

$\forall X_1, X_2, \dots, X_s \in \Gamma_0^1(M)$ and $Y_1, Y_2, \dots, Y_s \in \Gamma_1^0(M)$, where ϑ' is the adjoint operator of ϑ defined by;

$$(\vartheta' Y)(X) = Y(\vartheta X) = (Y \circ \vartheta)(X) \quad (8)$$

, $X \in \Gamma_0^1(M)$ and $Y \in \Gamma_1^0(M)$ [5].

A defined operator below;

$$\Phi_\vartheta: \Gamma_s^0(M) \rightarrow \Gamma_{s+1}^0(M)$$

is applied to the pure tensor field k of type $(0, s)$ with respect to ϑ which is given by;

$$\begin{aligned} (\Phi_\vartheta k)(X, Y_1, \dots, Y_s) &= (\vartheta X)k(Y_1, \dots, Y_s) - Xk(\vartheta Y_1, \dots, Y_s) \\ &+ \sum_{\lambda=1}^s k(Y_1, \dots, (L_{Y_\lambda} \vartheta)X, Y_s) \end{aligned} \quad (9)$$

for any $X, Y_1, \dots, Y_s \in \Gamma_0^1(M)$ where L_Y is the differentiation with respect to Y [5].

3 Integrable Silver Riemannian Structure

Let (M, ϑ, g) be a silver Riemannian manifold. The necessary and sufficient condition for a silver structure ϑ to be integrable, is to have a torsion-free affine connection ∇ with the structure tensor ϑ covariantly constant. Also, we know the integrability of ϑ is equivalent to the vanishing Nijenhuis tensor $N_\vartheta = 0$ [10].

We will now give another possible sufficient condition for integrability of a silver structure on Riemannian manifolds.

Theorem 3.1 Suppose (M, ϑ, g) be a silver Riemannian manifold. Then ϑ is

integrable if $(\Phi_{\vartheta}g) = 0$.

proof; from Def. 2.3 and taken $\nabla_g = 0$, we get;

$$g(X, (\nabla_Z \vartheta)Y) = g((\nabla_Z \vartheta)X, Y), \forall X, Y \in \Gamma_0^1(M) \quad (10)$$

Where ∇ is the Riemannian covariant derivative with respect to the metric g . Using Eq. 10 and $[X, Y] = \nabla_X Y - \nabla_Y X$, Eq. 9 can be written as;

$$\begin{aligned} (\Phi_{\vartheta}g)(X, Y_1, Y_2) = & -g((\nabla_X \vartheta)Y_1, Y_2) \\ & + g((\nabla_{Y_1} \vartheta)X, Y_2) + g(Y_1, (\nabla_X \vartheta)Y_2) \end{aligned} \quad (11)$$

also we can write,

$$\begin{aligned} (\Phi_{\vartheta}g)(Y_2, Y_1, X) = & -g((\nabla_{Y_2} \vartheta)Y_1, X) \\ & + g((\nabla_{Y_1} \vartheta)Y_2, X) + g(Y_1, (\nabla_X \vartheta)Y_2) \end{aligned} \quad (12)$$

by adding Eq.(11) and Eq.(12) we obtain

$$\begin{aligned} (\Phi_{\vartheta}g)(X, Y_1, Y_2) + (\Phi_{\vartheta}g)(Y_2, Y_1, X) = & -g((\nabla_X \vartheta)Y_1, Y_2) \\ & + g((\nabla_{Y_1} \vartheta)X, Y_2) + g(Y_1, (\nabla_X \vartheta)Y_2) \\ & - g((\nabla_{Y_2} \vartheta)Y_1, X) \\ & + g((\nabla_{Y_1} \vartheta)Y_2, X) + g(Y_1, (\nabla_X \vartheta)Y_2) \end{aligned} \quad (13)$$

but from (10) we can say

$$\begin{aligned} g((\nabla_X \vartheta)Y_1, Y_2) &= g(Y_1, (\nabla_X \vartheta)Y_2) \\ g((\nabla_{Y_1} \vartheta)Y_2, X) &= g(Y_2, (\nabla_{Y_1} \vartheta)X) \\ g((\nabla_{Y_2} \vartheta)Y_1, X) &= g(Y_1, (\nabla_{Y_2} \vartheta)X) \end{aligned} \quad (14)$$

by substituting Eq. (14) in Eq. (13) we get;

$$(\Phi_{\vartheta}g)(X, Y_1, Y_2) + (\Phi_{\vartheta}g)(Y_2, Y_1, X) = 2g(X, (\nabla_{Y_1} \vartheta)Y_2), \quad (15)$$

using $(\Phi_{\vartheta}g) = 0$ and Eq. (15) we get $(\nabla_{\vartheta}g) = 0$. Hence the proof.

Corollary 3.2 Suppose (M, ϑ, g) is a silver Riemannian manifold and ∇ is a levi-civita connection of g , if $(\Phi_{\vartheta}g) = 0$ we can equivalently say $\nabla_{\vartheta} = 0$.

Proposition 1 Suppose P is a corresponding almost product structure of a Silver Riemannian manifold (M, ϑ, g) . Then, the silver structure ϑ is integrable provided that $(\Phi_P g) = 0$

proof; from theorem (2.2) if $P = \frac{1}{\sqrt{2}}(\vartheta - I)$ is an almost product structure on M .

Conversely, P induces a silver structure on M as $\vartheta = I + \sqrt{2}P$. let $P^2 = I$ be an almost product structure on a Riemannian manifold (M, g) , then

$$\begin{aligned} \vartheta^2 &= (I + \sqrt{2}P)(I + \sqrt{2}P) \\ &= I + 2\sqrt{2}P + 2P^2 \\ &= I + 2\sqrt{2}P + 2I \\ &= I + 2(I + \sqrt{2}P) \\ &= I + 2\vartheta \end{aligned}$$

which satisfies Def.(2.1). Conversely,

$$\begin{aligned} P^2 &= \left(\frac{\vartheta - I}{\sqrt{2}}\right)^2 \\ &= \left(\frac{(\vartheta^2 - 2\vartheta) + I}{2}\right) \\ &= \frac{I + I}{2} = \frac{2I}{2} = I \end{aligned}$$

which prove our first assertion.

If a Riemannian metric g is pure with respect to a Silver structure ϑ , then the

Riemannian metric g is pure with respect to the corresponding almost product structure P . By using Eq. (2) and Eq. (13) we get;

$$(\Phi_P g) = \frac{2}{\sqrt{2}}(\Phi_\vartheta g) \quad (16)$$

In [5,21] it is shown that for an almost product Riemannian manifold with pure metric g , if $\Phi_P g = 0$, then the almost product structure P is integrable. Using theorem (3.1) and (16) we can have the following;

Proposition 2 Suppose P is a corresponding almost product structure of (M, ϑ, g) a silver Riemannian manifold, if $\Phi_P g = 0$, then the silver structure ϑ is said to be integrable.

Definition 3.3 Suppose (M, P, g) is a locally decomposable Riemannian manifold in which the Riemannian manifold (M, g) is endowed with an almost product structure $P^2 = I$. Such that for all vector fields X and Y , equation (4) and (5) holds, with also $\nabla_P = 0$ where ∇ is the covariant derivative. We see in [40], it was shown that $\nabla_P = 0$ is equivalent to decomposability of the pure metric g [35] p219-222 and [39] p418-420

Since $\Phi_P g = 0$ is the same as $\nabla_P = 0$, The following proposition can be stated.

Proposition 3 Suppose (M, ϑ, g) is a Silver Riemannian manifold. The manifold M is a locally decomposable Silver Riemannian manifold if and only if $\Phi_P g = 0$, where P is the corresponding almost product structure.

4 Twin Silver Riemannian Metric

Suppose (M, ϑ, g) is a silver Riemannian manifold. The twin Silver Riemannian metric can be defined by

$$G(X, Y) = (g \circ \vartheta)(X, Y) = g(\vartheta X, Y) = g(X, \vartheta Y) \quad (17)$$

$\forall$ vectors X and Y on M . We can show G as a new pure Riemannian metric using Def.(2.3).

$$\begin{aligned} G(\vartheta X, Y) &= (g \circ \vartheta)(\vartheta X, Y) \\ &= g(\vartheta(\vartheta X, Y)) = g(\vartheta^2 X, Y) \\ &= g((2\vartheta + I)X, Y) = 2g(\vartheta X, Y) + g(X, Y) \\ &= 2g(X, \vartheta Y) + g(X, Y) \\ &= g(X, (2\vartheta + I)Y) = g(X, \vartheta^2 Y) \\ &= g(X, \vartheta Y)\vartheta = (g \circ \vartheta)(X, \vartheta Y) \\ &= G(X, \vartheta Y) \end{aligned}$$

By apply the Φ_ϑ operator to the metric G . Using Eq.(9), we get

$$\begin{aligned} (\Phi_\vartheta G)(X, Y, Z) &= (\vartheta X)(G(Y, Z)) - X(G(\vartheta Y, Z)) + G((L_Y \vartheta)X, Z) + \\ &G(Y, (L_Z \vartheta)X) \\ &= (L_{\vartheta X} G - L_X(G \circ \vartheta))(Y, Z) + G(Y, \vartheta L_X Z) - G(\vartheta Y, L_X Z) \end{aligned}$$

Using Eq.(17)

$$\begin{aligned} (\Phi_\vartheta G)(X, Y, Z) &= (L_{\vartheta X}(g \circ \vartheta) - L_X((g \circ \vartheta) \circ \vartheta))(Y, Z) + (g \circ \vartheta)(Y, \vartheta L_X Z) - (g \circ \vartheta)(\vartheta Y, L_X Z) \\ &= (L_{\vartheta X} g) \circ \vartheta + g \circ (L_{\vartheta X} \vartheta) - L_X(g \circ \vartheta) \circ \vartheta - (g \circ \vartheta)L_X \vartheta(Y, Z) \\ &+ (g \circ \vartheta)(Y, \vartheta L_X Z) - (g \circ \vartheta)(\vartheta Y, L_X Z) \\ &= (L_{\vartheta X} g - L_X(g \circ \vartheta))(\vartheta Y, Z) + g(\vartheta Y, \vartheta L_X Z) - g(\vartheta(\vartheta Y), L_X Z) \\ &+ (g \circ L_{\vartheta X} Z - (g \circ \vartheta)L_X Z)(Y, Z) \\ &= (\Phi_{\vartheta X} g)(X, \vartheta Y, Z) + g(L_{\vartheta X} \vartheta - g(\vartheta(L_X \vartheta)Y, Z)) \\ &= (\Phi_{\vartheta X} g)(X, \vartheta Y, Z) + g([\vartheta X, \vartheta Y] - \vartheta[\vartheta X, Y], Z) \end{aligned}$$

$$\begin{aligned}
 & +g(\vartheta[X, \vartheta Y] - \vartheta^2[X, Y], Z) \\
 & = (\Phi_{\vartheta X} g)(X, \vartheta Y, Z) + g([\vartheta X, \vartheta Y] - \vartheta[\vartheta X, Y] \\
 & - \vartheta[X, \vartheta Y] + \vartheta^2[X, Y], Z) \\
 & = (\Phi_{\vartheta X} g)(X, \vartheta Y, Z) + g(N_{\vartheta}(X, Y), Z)
 \end{aligned}$$

we can now have the theorem

Theorem 4.1 For a given silver Riemannian manifold (M, ϑ, g) , we have the following

$$\Phi_{\vartheta} G = \Phi_{\vartheta X} g \circ \vartheta + g \circ N_{\vartheta} \quad (18)$$

Corollary 4.2 For a given Silver Riemannian manifold (M, ϑ, g) the following conditions are equivalent.

- $\Phi_{\vartheta} G = 0$
- $\Phi_{\vartheta} g = 0$

Theorem 4.3 Suppose (M, ϑ, g) is a locally decomposable Silver Riemannian manifold. Then the levi-civita connection of the silver Riemannian metric g and the levi-civita connection of the twin Silver Riemannian metric G coincides to each other.

proof; Let ${}^g \nabla$ be a covariant derivative of the levi-civita connection of the Silver Riemannian metric g . BY using theorem (3.1) and (4.1), we get;

$$\begin{aligned}
 {}^g \nabla_G &= ({}^g \nabla_g) \circ \vartheta + g \circ ({}^g \nabla_{\vartheta}) \\
 &= g \circ ({}^g \nabla_{\vartheta}) = 0
 \end{aligned}$$

this shows ${}^g \nabla_G = 0$ and also by proposition (1) ${}^g \nabla_P = 0$.

5 Curvature Properties of locally decomposable Silver Riemannian manifold

If a pure tensor k satisfies $\Phi_P k = 0$, then it is called Φ – tensor. If the $(1,1)$ tensor P is a complex structure, then Φ is an analytic tensor. If P is an almost product structure such that the Nijenhuis tensor vanishes, then Φ is a decomposable tensor [29].

Suppose R and N are the curvature tensors derived by the silver Riemannian metric g and the the twin Silver Riemannian metric G respectively. According to theorem (4.3), for a locally decomposable Silver Riemannian manifold we have $R = N$. Since the Riemannian curvature tensor R is pure with respect to an almost product structure [5], therefore using theorem (2.2) we can also say R is pure with respect to the Silver structure ϑ , by applying the Φ – operator to R . Using similar way in proof of theorem (3.1). We can show that;

$$(\Phi_{\vartheta} R)(X, Y_1, Y_2, Y_3, Y_4) = (\nabla_{\vartheta X} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla_X R)(\vartheta Y_1, Y_2, Y_3, Y_4) \quad (19)$$

Taking into account R is pure and applying the Bianchi's 2nd identity to Eq.(19), we get;

$$\begin{aligned}
 (\Phi_{\vartheta} R)(X, Y_1, Y_2, Y_3, Y_4) &= g((\nabla_{\vartheta X} R)(Y_1, Y_2, Y_3) - (\nabla_X R)(\vartheta Y_1, Y_2, Y_3, Y_4)) \\
 &= g((\nabla_{\vartheta X} R)(Y_1, Y_2, Y_3) - \vartheta(\nabla_X R)(Y_1, Y_2, Y_3, Y_4)) \\
 &= g(-(\nabla_{Y_1} R)(Y_2, \vartheta X, Y_3) - (\nabla_{Y_2} R)(\vartheta X, Y_1, Y_3) \\
 &\quad - \vartheta(\nabla_X R)(Y_1, Y_2, Y_3, Y_4))) \quad (20)
 \end{aligned}$$

On the other hand taking $\nabla_{\vartheta} = 0$ we get;

$$\begin{aligned}
 (\nabla_{Y_2} R)(\vartheta X, Y_1, Y_3) &= \nabla_{Y_2} (R(\vartheta X, Y_1, Y_3)) - R(\nabla_{Y_2} (\vartheta X), Y_1, Y_3) - \\
 R(\vartheta X, \nabla_{Y_2} Y_1, Y_3) &\quad - R(\vartheta X, Y_1, \nabla_{Y_2} Y_3) \\
 &= \\
 (\nabla_{Y_2} \vartheta)(R(X, Y_1, Y_3)) &+ \vartheta(\nabla_{Y_2} R(X, Y_1, Y_3)) - R((\nabla_{Y_2} \vartheta)X + \vartheta(\nabla_{Y_2} X), Y_1, Y_3) \\
 &\quad - R(\vartheta X, \nabla_{Y_2} Y_1, Y_3) - R(\vartheta X, Y_1, \nabla_{Y_2} Y_3)
 \end{aligned}$$

$$= \vartheta(\nabla_{Y_2} R(X, Y_1, Y_3)) - \vartheta(R(\nabla_{Y_2} X, Y_1, Y_3)) - \vartheta(R(X, \nabla_{Y_2} Y_1, Y_3)) \\ - \vartheta(R(X, Y_1, \nabla_{Y_2} Y_3))$$

therefore,

$$(\nabla_{Y_2} R)(\vartheta X, Y_1, Y_3) = \vartheta((\nabla_{Y_2} R)(X, Y_1, Y_3)) \quad (21)$$

In similar way we can find;

$$(\nabla_{Y_1} R)(Y_2, \vartheta X, Y_3) = \vartheta((\nabla_{Y_1} R)(Y_2, X, Y_3)) \quad (22)$$

By using Eq. (21) and (22) into Eq.(20) we get;

$$(\Phi_{\vartheta} R)(X, Y_1, Y_2, Y_3, Y_4) = g(-\vartheta(\nabla_{Y_2} R)(Y_2, X, Y_3) - \vartheta(\nabla_{Y_2} R)(X, Y_1, Y_3) \\ - \vartheta(\nabla_X R)(Y_1, Y_2, Y_3, Y_4))$$

$$= -g(\vartheta\{\zeta(\nabla_X R)(Y_1, Y_2)\}, Y_3, Y_4) = 0$$

Where ζ denotes the cyclic sum with respect to X, Y_1 and Y_2 . We can then have the following;

Theorem 5.1 *For a given locally decomposable Silver Riemannian manifold, the Riemannian curvature tensor field is a Φ – tensor field.*

By using Eq.(2) and Eq.(9) we get;

$$(\Phi_P R) = \frac{2}{\sqrt{2}}(\Phi_{\vartheta} R) \quad (23)$$

Where ϑ is a Silver structure and P is its corresponding almost product structure. From Eq.(23) we can have the following;

Proposition 4 *In a locally decomposable Silver Riemannian manifold, the Riemannian curvature tensor field is a decomposable tensor field.*

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