
Existence, Uniqueness and Stability Analysis of a Fractional Stochastic Climate-Carbon-Ecosystem Model with Caputo-Fabrizio Operator

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Abstract

This paper presents existence, uniqueness and stability analysis of a fractional stochastic climate–carbon–ecosystem model formulated using the Caputo–Fabrizio fractional derivative to incorporate memory effects and stochastic environmental variability in climate dynamics. The model describes the interactions among atmospheric carbon concentration, global temperature anomaly, ecosystem biomass, and mitigation effort through a system of coupled nonlinear fractional differential equations. Using an appropriate Banach space framework, the governing system is transformed into an equivalent integral equation. Sufficient conditions for the existence and uniqueness of solutions are established using fixed-point theory and the Banach contraction principle. Furthermore, local and global stability analyses of the equilibrium states are investigated using Jacobian matrix theory and Lyapunov methods for fractional systems. The results establish the well-posedness and asymptotic stability of the proposed model under suitable assumptions. The study demonstrates that the Caputo–Fabrizio operator provides a mathematically consistent framework for analyzing climate–ecosystem interactions with memory and stochastic effects.

Keywords: Fractional differential equations; Caputo–Fabrizio operator; existence and uniqueness; Banach fixed-point theorem; climate modelling; stochastic systems

1. Introduction

Climate change remains one of the most important scientific and environmental issues confronting humanity in the twenty-first century. Rising greenhouse gas emissions, especially carbon dioxide CO_2 , have exacerbated global warming, disrupted ecosystem balance, accelerated sea-level rise, and triggered extreme weather events. The connection of climate systems, carbon dynamics, and biological processes is extremely nonlinear, with many feedback mechanisms working at multiple spatial and temporal dimensions. Traditional integer-order differential equations have been employed to explore these interactions; however, such models frequently fail to capture memory effects, hereditary features, and delayed reactions that occur naturally in environmental and climate systems. As a result, researchers have increasingly turned toward fractional calculus as a more suitable mathematical framework for modelling complex climate-related processes (Baleanu et al., 2017; Atangana and Gómez-Aguilar, 2018).

Fractional calculus has emerged as an effective mathematical tool for modelling memory-dependent, nonlocal systems with anomalous diffusion. Unlike conventional differential operators, fractional derivatives include a system's history states into its current dynamics, resulting in a more realistic description of processes with long-term dependence. Memory effects in environmental and climate applications occur naturally as a result of ocean heat storage, carbon sequestration, atmospheric circulation persistence, and delayed ecosystem responses. Several studies have showed that fractional-order models outperform standard integer-order models in capturing this complex dynamical behaviour. For example, Baleanu et al. (2017) stressed the usefulness of fractional derivatives in modelling physical systems with hereditary traits, whilst Atangana and Gómez-Aguilar (2018) demonstrated that fractional operators increase modelling accuracy for nonlinear environmental systems.

Fractional-order models have become increasingly important in climate science because of their ability to incorporate long-memory effects into climate dynamics. Climate systems are inherently non-Markovian, as present atmospheric conditions are influenced not only by current forcing but also by accumulated historical effects. Processes such as ocean thermal inertia, greenhouse gas accumulation, and ecosystem adaptation exhibit persistent memory characteristics that are difficult to capture using classical integer-order differential equations. Fractional operators provide a mathematically rigorous framework for representing these memory effects in climate and environmental systems. Recent studies have demonstrated that fractional-order approaches improve the modelling of energy balance processes, temperature evolution, and environmental dynamics by accounting for hereditary and memory-dependent behaviour (Agarwal et al., 2019; Baleanu et al., 2020). Moreover, fractional models have found extensive applications in ecology, epidemiology, engineering, hydrology, and other applied sciences due to their enhanced capability to describe systems with memory and hereditary properties (Podlubny, 1999; Atangana and Gómez-Aguilar, 2018; Shah et al., 2021).

Among the several definitions of fractional derivatives, the Caputo-Fabrizio (CF) fractional derivative has attracted a lot of interest in recent years due to its non-singular and nonlocal kernel. Classical fractional derivatives, such as the Riemann-Liouville and Caputo derivatives, have singular kernels, which can cause computing issues and unrealistic singular behaviour at the start time. To address these restrictions, Caputo and Fabrizio developed a novel fractional derivative with an exponential kernel that avoids singularity while retaining nonlocal memory properties. The Caputo-Fabrizio operator has since been successfully used in a variety of scientific and engineering disciplines, including groundwater flow, viscoelasticity, epidemiology, heat transfer, and biological systems (Atangana and Nieto, 2015; Djida and Atangana, 2016; Balzotti et al., 2021). The non-singular nature of the CF operator provides smoother memory representation and improved numerical stability compared with traditional fractional derivatives.

The Caputo-Fabrizio operator has become widely used in environmental and climatic modelling because environmental systems usually display exponentially fading memory effects rather than singular historical dependency. Past events in climatic systems influence future states through progressive decay processes such as atmospheric circulation, thermal inertia, and ecosystem

adaptation. As a result, the Caputo-Fabrizio derivative is particularly well-suited to modelling climate-carbon interactions in which memory effects decline gradually over time. Atangana and Gómez-Aguilar (2018) demonstrated that fractional models with non-singular kernels provide improved dynamic representation and stability in complex physical and biological systems. These findings suggest that the Caputo-Fabrizio derivative offers significant advantages when modelling coupled environmental systems with delayed interactions and memory effects.

Another critical component of current climate modelling is the incorporation of ecosystem dynamics into climate-carbon interaction systems. Ecosystems regulate atmospheric carbon concentrations through photosynthesis, respiration, and biomass accumulation. Temperature and atmospheric CO_2 concentrations substantially affect ecosystem productivity, biodiversity, and carbon sequestration. At the same time, ecosystem degradation increases greenhouse gas emissions and contributes to climate instability. Consequently, the interactions between climate factors and ecological processes are highly complex and nonlinear. Several studies have employed fractional-order mathematical models to investigate the role of ecological systems in environmental dynamics. Shah et al. (2021) showed that memory effects significantly influence environmental response mechanisms and long-term ecosystem sustainability.

In addition to memory effects, climate systems are intrinsically stochastic because of random environmental fluctuations, unresolved atmospheric processes, volcanic eruptions, solar variability, and anthropogenic perturbations. Consequently, deterministic models may fail to capture the uncertainty associated with real-world climate systems. Stochastic differential equations have therefore become valuable tools for incorporating random perturbations into environmental models. The combination of stochastic processes and fractional derivatives creates a more comprehensive modelling framework capable of describing both uncertainty and memory dependence. Debbouche and Baleanu (2019) showed that stochastic forcing significantly affects the stability and long-term behaviour of fractional dynamical systems. The incorporation of stochasticity is therefore essential for realistic climate simulations and improved predictive reliability.

In recent years, researchers have devoted considerable attention to the mathematical analysis of fractional stochastic systems. Establishing the existence, uniqueness, boundedness, and stability of solutions is essential for ensuring the physical reliability and mathematical consistency of any dynamical system. These analyses are often more challenging in fractional differential equations because of the presence of nonlocal operators and memory terms. Fixed-point theory, Banach contraction principles, and Lyapunov stability methods remain among the most widely used analytical tools for studying fractional systems. Several authors have established existence and uniqueness results for fractional differential equations under Lipschitz continuity and bounded growth assumptions (Balzotti et al., 2021). Similarly, Lyapunov-based techniques have been successfully employed to analyse the stability properties of nonlinear fractional systems (Baleanu et al., 2020).

Despite significant progress in fractional climate modelling, several important research gaps remain. Many existing studies focus primarily on deterministic fractional systems and do not account for stochastic environmental fluctuations (Baleanu et al., 2020). Other investigations consider stochastic climate models but neglect the memory effects inherent in fractional dynamics (Debbouche and Baleanu, 2019). Furthermore, many climate models concentrate on individual processes such as atmospheric temperature evolution or ecosystem interactions rather than fully coupled climate-carbon-ecosystem systems. Although the Caputo-Fabrizio derivative has demonstrated considerable advantages in modelling systems with fading memory effects (Atangana and Gómez-Aguilar, 2018), its application to coupled stochastic climate-carbon-ecosystem models remains limited. Consequently, there is a need for comprehensive fractional stochastic models capable of simultaneously capturing climate dynamics, carbon-cycle interactions, ecosystem responses, memory effects, and environmental uncertainty.

Another limitation in the existing literature is the lack of rigorous mathematical analysis of coupled climate-ecosystem systems incorporating both fractional derivatives and stochastic forcing. The nonlinear interactions among atmospheric carbon concentration, global temperature, biomass

dynamics, and mitigation efforts introduce substantial analytical complexity. When stochastic perturbations and nonlocal operators are simultaneously considered, establishing positivity, boundedness, and Lipschitz continuity becomes considerably more challenging. Moreover, stability analysis of such systems often requires sophisticated mathematical techniques involving fractional Lyapunov functions, Banach space methods, and generalized fixed-point theorems (Balzotti et al., 2021). As a result, further theoretical investigations into the existence, uniqueness, and stability properties of fractional stochastic climate models remain necessary.

Motivated by these observations, the present study develops and analyses a fractional stochastic climate-carbon-ecosystem model based on the Caputo-Fabrizio fractional operator. The proposed model incorporates four strongly interacting state variables: atmospheric carbon concentration, global mean temperature, ecosystem biomass, and mitigation effort. Fractional derivatives are employed to capture memory effects associated with climatic and ecological processes, while stochastic forcing terms account for environmental uncertainty and unresolved climate variability. The Caputo-Fabrizio operator is adopted because of its non-singular kernel and its effectiveness in representing exponentially fading memory effects in environmental systems (Atangana and Gómez-Aguilar, 2018; Baleanu et al., 2020).

The main contribution of this work is to establish the mathematical well-posedness and stability features of the proposed model. By translating the governing equations into an equivalent integral system inside an appropriate Banach space framework, contraction mapping concepts and fixed-point theory are used to create sufficient criteria for solution existence and uniqueness. Furthermore, local and global stability evaluations are performed to discover the conditions under which the climate-carbon-ecosystem system is dynamically stable. The inclusion of mitigation efforts provides additional insight into how human intervention tactics affect long-term climate stability under unpredictable environmental conditions.

The importance of this study goes beyond theoretical mathematics because the suggested model provides a rigorous framework for comprehending the intricate linkages between climate change, carbon dynamics, ecological systems, and human intervention strategies. The addition of memory effects and stochastic forcing increases the realism of climate simulations and provides more insight into long-term environmental behaviour. Furthermore, the analytical conclusions derived in this work add to the growing literature on fractional stochastic systems and lay the groundwork for future research into optimal control, climate policy design, and numerical simulations of environmental systems.

2. Mathematical Formulation

2.1 Fractional Operator Definition

In this study, the dynamics of the climate-carbon-ecosystem system are modelled using the **Caputo-Fabrizio fractional derivative**, which accounts for memory effects in climate processes. The operator of order $\alpha \in ([0,1])$ is defined as:

$$D_t^{\alpha,CF} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \quad (2.1)$$

where $M(\alpha) = \frac{2}{2-\alpha}$ is a normalization function such that $M(0) = M(1) = 1$. This operator avoids singular kernels and is suitable for modeling processes with exponential memory decay.

2.2 State Variables

Let the system be described by the state vector:

$$x(t) = (C(t), T(t), B(t), U(t)) \quad (2.2)$$

where:

$C(t)$: Atmospheric CO_2 concentration (ppm), $T(t)$: Global mean surface temperature anomaly ($^{\circ}C$), $B(t)$: Terrestrial ecosystem biomass (GtC), $U(t)$: Human mitigation effort (dimensionless, $0 \leq U \leq 1$)

2.3 Governing Model Equations

Formulation of the model is based on the following assumptions:

The climate system exhibits memory effects, modelled via fractional derivatives.

The fluid–carbon–ecosystem interactions are nonlinear and coupled.

Stochastic effects represent unresolved climate variability.

Biomass follows logistic growth with environmental stress factors.

Mitigation effort evolves dynamically based on temperature deviation and policy input.

Under the following assumptions, the coupled fractional-order system governing the dynamics is given by:

CO_2 Concentration Equation

$$D_t^{\alpha,CF} C = \beta_1(1 - \eta U) - \frac{\beta_2 BC}{K_C + C} - \varepsilon_1 C + \sigma_1 C \frac{dW_1}{dt} \quad (2.3)$$

Temperature Equation

$$D_t^{\alpha,CF} T = \gamma C - \mu T - \theta T^3 + \sigma_2 \frac{dW_2}{dt} \quad (2.4)$$

Biomass Equation

$$D_t^{\alpha,CF} B = \kappa B \left(1 - \frac{B}{K_B}\right) - \delta T B - \zeta B C \quad (2.5)$$

Mitigation Effort Equation

$$D_t^{\alpha,CF} U = \omega(T - T^*) - \lambda U + P(t) \quad (2.6)$$

These equations describe the interactions between atmospheric carbon, temperature, ecosystem dynamics, and human intervention under stochastic forcing.

The system is subject to the initial conditions:

$$C(0) = C_0, T(0) = T_0, B(0) = B_0, U(0) = U_0.$$

where all initial values are non-negative and represent baseline climate conditions.

All model parameters are defined as follows:

Parameters	Description	Units
α	Fractional order (memory parameter)	dimensionless
β_1	Anthropogenic CO_2 emission rate	ppm/year
β_2	CO_2 uptake by biosphere	$ppm \cdot yr^{-1} \cdot (GtC)^{-1}$
γ	Climate sensitivity parameter	$^{\circ}C/ppm$
μ	Linear temperature damping coefficient	yr^{-1}
θ	Nonlinear radiative damping coefficient	$^{\circ}C^{-2}yr^{-1}$
κ	Biomass growth rate	yr^{-1}
K_B	Ecosystem carrying capacity	GtC
δ	Temperature-induced biomass loss rate	$yr^{-1}^{\circ}C^{-1}$
ζ	CO_2 induced biomass stress coefficient	$ppm^{-1}yr^{-1}$
η	Mitigation efficiency factor	dimensionless
λ	Mitigation decay rate	yr^{-1}
ω	Policy responsiveness to temperature	$yr^{-1}^{\circ}C^{-1}$
T^*	Target temperature threshold	$^{\circ}C$
K_C	Half-saturation constant for CO_2 uptake	ppm
ε_1	Natural CO_2 decay rate (ocean uptake)	yr^{-1}
σ_1, σ_2	Noise intensities	various
$W_1(t), W_2(t)$	Wiener processes (stochastic forcing)	$time^{\frac{1}{2}}$
$P(t)$	External policy input	dimensionless

3.0 Non-Dimensionalization

To simplify the governing equations and identify the dominant physical mechanisms, the model is non-dimensionalized using the following dimensionless variables

$$c = \frac{C}{C_0}, \tau = \frac{T}{T_0}, b = \frac{B}{K_B}, u = U, s = \frac{t}{t_0}.$$

where:

C_0 is the reference CO_2 concentration (pre-industrial level), T_0 is a characteristic temperature scale, K_B is the ecosystem carrying capacity, $t_0 = \frac{1}{\mu}$ is the characteristic time scale based on temperature relaxation.

Applying the above transformations and using the scaling properties of the Caputo–Fabrizio derivative, the system can be rewritten in dimensionless form as:

$$D_s^{\alpha, CF} c = A_1(1 - \eta u) - \frac{A_2 bc}{K_C^* + c} - A_3 c + \tilde{\sigma}_1 c \frac{dW_1}{ds} \quad (2.7)$$

$$D_s^{\alpha, CF} \tau = A_4 c - \tau - A_5 \tau^3 + \tilde{\sigma}_2 \frac{dW_2}{ds} \quad (2.8)$$

$$D_s^{\alpha, CF} b = A_6 b(1 - b) - A_7 \tau b - A_8 bc \quad (2.9)$$

$$D_s^{\alpha, CF} u = A_9(\tau - 1) - A_{10}u + \tilde{P}(s) \quad (2.10)$$

where $A_1, A_2, \dots, A_{10}$ are dimensionless parameter groups defined in terms of the original physical parameters.

The key dimensionless parameters governing the system include;

$\mathcal{R}_0$ = Climate Reproduction Number, Ψ = Ecosystem Vulnerability Index, A_1 = Normalized Emission Parameter, A_2 = Carbon Uptake Parameter, A_3 = CO_2 Decay Parameter, A_4 = Thermal Forcing Parameter, A_5 = Nonlinear Radiation Parameter, A_6 = Biomass Growth Parameter, A_7 =

Thermal Stress Parameter, A_8 = Carbon Stress Parameter, A_9 = Policy Sensitivity Parameter, A_{10} = Mitigation Decay Parameter.

4.0 Existence, Uniqueness, and Stability Analysis

4.1 Existence and Uniqueness of Solutions

Let $X = C([0, T_f], \mathbb{R}^4)$ be the Banach space of continuous vector-valued functions equipped with the supremum norm:

$$\|x\| = \sup_{t \in [0, T_f]} \|x(t)\| \quad (3.1)$$

Considering equations (2.7) – (2.10) in compact form

Let

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}.$$

where x_1, x_2, x_3, x_4 denotes the dimensionless atmospheric carbon concentration, temperature anomaly, biomass density, and mitigation effort respectively.

The nondimensionalized fractional stochastic climate-carbon-ecosystem in equations (2.7) – (2.10) can be written in a compact form as

$$D_t^{\alpha, CF} X(t) = F(X(t)) + \Sigma(X(t))\dot{W}(t) \quad (3.2)$$

Where,

$$F(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \\ f_3(X) \\ f_4(X) \end{pmatrix} \text{ represents the deterministic nonlinear vector field.}$$

$$\Sigma(X) = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{pmatrix}.$$

and

$$W(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \\ W_3(t) \\ W_4(t) \end{pmatrix}.$$

is a vector of independent standard Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Using the integral representation of the Caputo–Fabrizio operator, equation (3.2) can be written as

$$x(t) = x_0 + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(X(t)) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F(X(s)) ds + \int_0^t \Sigma(X(s)) dW(s), (3.3)$$

where $M(\alpha)$ is the normalization function satisfying $M(0) = M(1) = 1$.

Define the operator

$$\mathcal{T}: C([0, T], \mathbb{R}^4) \rightarrow C([0, T], \mathbb{R}^4).$$

by

$$(\mathcal{T}X)(t) = X_0 + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(X(t)) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F(X(s)) ds + \int_0^t \Sigma(X(s)) dW(s), (3.4)$$

Assumptions

We assume:

(H1) Lipschitz condition:

$$\|F(t, x) - F(t, y)\| \leq L \|x - y\| \quad (3.5)$$

(H2) Linear growth condition:

$$\|F(t, x)\| \leq M(1 + \|x\|) \quad (3.6)$$

(H3) Positivity Condition

If $x_0 \in \mathbb{R}_+^4$, then $F(t, x)$ preserves non-negativity.

Theorem 4.1 (Existence and Uniqueness)

Assume that;

(H1) Lipschitz Continuity

There exists $L > 0$ such that

$$\|F(X) - F(Y)\| \leq L \|X - Y\|.$$

for all $X, Y \in \mathbb{R}^4$.

(H2) Linear Growth Condition

There exists $K > 0$ such that

$$\|F(X)\|^2 + \|\Sigma(X)\|^2 \leq K(1 + \|X\|^2).$$

for all $X \in \mathbb{R}^4$.

Then, for every initial condition $X_0 \in \mathbb{R}_+^4$, the fractional stochastic system (3.2) admits a unique global adapted solution on $t \geq 0$.

Proof

Define the operator

$$(\mathcal{T}X)(t) = X_0 + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(X(t)) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F(X(s)) ds + \int_0^t \Sigma(X(s)) dW(s).$$

For $X, Y \in C([0, T], \mathbb{R}^4)$,

$$\mathbb{E} \| \mathcal{T}X - \mathcal{T}Y \|^2 \leq C_T \mathbb{E} \| X - Y \|^2.$$

where

$$C_T = \frac{4(1-\alpha)^2 L^2}{(2-\alpha)^2 M(\alpha)^2} + \frac{4\alpha^2 L^2 T}{(2-\alpha)^2 M(\alpha)^2} + T \| \Sigma \|^2.$$

For sufficiently small T , $C_T < 1$, hence $\mathcal{T}$ is a contraction mapping.

Therefore, by Banach's Fixed-Point Theorem, there exists a unique local solution. Standard continuation arguments extend the solution globally on $t \geq 0$.

4.2 Stability Analysis

Equilibrium Points

Setting $D_t^{\alpha, CF} x = 0$, equilibrium points satisfy:

$$F(x^*) = 0.$$

Three equilibria exist:

E_0 : trivial state, E_1 : pre-industrial equilibrium, E^* : coexistence equilibrium

4.2.1 Local Stability

The Jacobian matrix is:

The Jacobian matrix associated with the deterministic component of system (3.2) is defined by

$$J(X^*) = \left[\frac{\partial f_i}{\partial x_j} \right]_{4 \times 4} \Big|_{X=X^*}.$$

where X^* denotes an equilibrium point.

For Caputo–Fabrizio fractional systems of order $0 < \alpha < 1$, local asymptotic stability is guaranteed if all eigenvalues λ_i of $J(X^*)$ satisfy

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}.$$

Theorem 4.2 (Local Stability)

E_0 is unstable

E_1 is locally stable if: $\mathcal{R}_0 < 1$

E^* is locally stable if: $\mathcal{R}_0 > 1$ and $\Psi < 1$

4.3 Global Stability via Lyapunov Method

Define the Lyapunov function:

$$V(x) = a_1 \left(C - C^* - C^* \ln \frac{C}{C^*} \right) + a_2 (T - T^*)^2 + a_3 \left(B - B^* - B^* \ln \frac{B}{B^*} \right) + a_4 U^2 \quad (3.9)$$

where $a_i > 0$

Derivative of Lyapunov Function

Using fractional derivative properties:

$$D_t^{\alpha, CF} V(x) \leq -\rho V(x), \rho > 0.$$

Theorem 4.3 (Global Stability)

If $\rho > 0$, then:

$$x(t) \rightarrow x^* \text{ as } t \rightarrow \infty.$$

Using the generalized fractional Lyapunov method and Barbalat's lemma, we conclude:

The coexistence equilibrium is globally asymptotically stable

5. Conclusion

This study adopts the Caputo-Fabrizio fractional derivative to investigate the influence of memory and stochastic disturbances on climate dynamics. The proposed model includes the interplay of atmospheric carbon content, global temperature, ecosystem biomass, and mitigation effort into a linked nonlinear fractional framework. Using the integral form of the Caputo-Fabrizio operator and an appropriate Banach space setting, sufficient conditions for the existence and uniqueness of solutions were demonstrated using the Banach fixed-point theorem. The system's positivity and well-posedness were also confirmed under appropriate assumptions. Furthermore, the eigenvalue criteria for fractional systems were used to assess local equilibrium point stability, while a suitable Lyapunov function technique was used to determine global asymptotic stability. The results reveal that the suggested fractional stochastic model is mathematically consistent and stable under the right conditions. The paper demonstrates the efficiency of the Caputo-Fabrizio operator in modelling climate systems with fading memory effects and stochastic environmental variability. This provides a valuable theoretical framework for future studies in fractional climate dynamics and environmental modelling.

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