
Fixed-Point Analysis, Stability, and Computational Well-Posedness of a Fractional Stochastic Magnetoconvection Model in a Porous Vertical Annulus

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Abstract

This study investigates a fractional stochastic magnetoconvection model describing the natural convection flow of an electrically conducting fluid in a porous vertical annulus with variable thermal conductivity. The governing momentum and energy equations are formulated using the Caputo–Fabrizio fractional derivative to incorporate memory effects, while stochastic perturbations are introduced to account for random environmental and thermal fluctuations. Rather than seeking analytical solutions, the study focuses on establishing the mathematical validity and computational admissibility of the proposed model. The governing system is reformulated as an equivalent system of fractional integral equations, and an associated solution operator is constructed within an appropriate Banach space. The boundedness and Lipschitz continuity of the nonlinear operators are established, and fixed-point theory is employed to investigate the existence and uniqueness of solutions. Using the Banach Fixed Point Theorem, sufficient conditions guaranteeing a unique solution are obtained. The results further demonstrate that the model is well-posed and suitable for reliable numerical computation and future simulation studies.

Keywords: Fractional calculus, Caputo–Fabrizio derivative, fixed-point theorem, stochastic magnetoconvection, porous vertical annulus, existence and uniqueness, well-posedness.

1. Introduction

Magnetohydrodynamic (MHD) convection flows in porous media have sparked widespread interest due to its extensive applications in geothermal systems, petroleum recovery, nuclear reactor cooling, and metallurgical processes. According to Nield and Bejan (2017), the interaction of magnetic fields with electrically conducting fluids is critical in determining momentum and heat transfer properties. Natural convection in annular geometries has garnered a lot of interest because of its applications in heat exchangers, thermal energy storage systems, and double-pipe engineering devices. Recent studies by Sheikholeslami (2018) and Chamkha et al. (2020) shown that magnetic field influences can significantly alter thermal transport pathways in porous media.

The difficulties of traditional integer-order differential models in modeling memory-dependent transport processes prompted the creation of fractional calculus models. Podlubny (2011) stated that fractional derivatives give a suitable framework for modeling hereditary phenomena found in many physical systems. In fluid mechanics and heat transfer applications, fractional models have been proven to better reflect anomalous diffusion and long-term memory effects than traditional models. Baleanu, Machado, and Luo (2017) found that fractional operators improve physical realism in a variety of engineering applications, while Atangana and Gómez-Aguilar (2018) demonstrated their efficacy in modeling complicated transportation phenomena.

Among the several fractional operators proposed in the literature, the Caputo-Fabrizio derivative proposed by Caputo and Fabrizio (2015) has emerged as a significant tool due to its nonsingular exponential kernel. Unlike classic Caputo and Riemann-Liouville operators, the Caputo-Fabrizio derivative removes kernel singularities while retaining memory effects. Losada and Nieto (2015) demonstrated various fundamental mathematical features of this operator, while Atangana and Baleanu (2016) effectively applied it to heat transfer and diffusion issues. As a result, the operator has grown in popularity for modelling fluid flow and thermal systems that incorporate memory effects.

Realistic fluid systems are frequently prone to uncertainties caused by environmental changes, thermal disturbances, and measurement errors. To account for these effects, stochastic differential equations are commonly used in mathematical modelling. Kloeden and Platen (2011) established a complete framework for stochastic dynamical systems, while Lord, Powell, and Shardlow (2014) proved the significance of stochastic formulations in computational fluid dynamics. The introduction of stochastic forcing elements allows mathematical models to better describe physical systems that operate under uncertain environmental conditions.

Recent research have demonstrated the benefits of integrating fractional calculus and stochastic analysis. Atangana and Alkahtani (2016) shown that fractional stochastic models improve representations of complex physical processes involving both uncertainty and memory effects. Similarly, Baleanu et al. (2018) found that fractional stochastic formulations can capture dynamic characteristics that are difficult to characterize using strictly deterministic methods. Khan et al. (2020) used fractional operators to study transport phenomena and showed that they are effective at representing nonlocal fluid dynamics.

The impact of varying thermal conductivity on convective transport has also sparked considerable study attention. Makinde (2012) noted that assuming constant thermal conductivity can result in erroneous thermal forecasts in a variety of engineering applications. Ellahi et al. (2017) and Sheikholeslami and Bhatti (2018) found that temperature-dependent thermal conductivity has a significant impact on heat transfer rates and thermal boundary layer growth. Thus, including changing thermal conductivity into mathematical models is critical for producing realistic simulations of heat transport processes.

Despite these gains, few research have been conducted to rigorously validate fractional stochastic magnetoconvection models in porous annular geometries. Most present efforts stress analytical approximations or numerical simulations without confirming essential features such as existence, uniqueness, and stability of solutions. However, Zeidler (2013) stressed that the dependability of any computing approach is determined by the mathematical model's well-posedness. Similarly, Granas and Dugundji (2013) and Smart (2014) emphasized the significance of fixed-point theory in establishing existence and uniqueness conclusions for nonlinear differential equations.

Motivated by the foregoing observations, the present study investigates a fractional stochastic magnetoconvection model describing natural convection flow of an electrically conducting fluid in a porous vertical annulus with variable thermal conductivity. The governing equations are formulated using the Caputo–Fabrizio fractional derivative to incorporate memory effects, while stochastic perturbations are introduced to model random environmental and thermal fluctuations. By reformulating the governing equations as equivalent fractional integral equations and constructing an appropriate solution operator in a Banach space, fixed-point theory is employed to establish boundedness, Lipschitz continuity, existence, uniqueness, and computational well-posedness of the model.

2. Mathematical Formulation

In formulating a mathematical model for unsteady fractional stochastic magnetohydrodynamics (MHD) natural convection flow of an electrically conducting viscous fluid in a porous vertical annulus with variable thermal conductivity, consider an incompressible electrically Newtonian fluid confined within the annular region formed by two infinitely long concentric vertical cylinders of radii a and b , where $b > a$. The annular region is saturated with a homogeneous and isotropic porous medium. A uniform magnetic field of strength B_0 is applied radially inward, normal to the flow direction.

The inner cylinder is maintained at a constant higher temperature T_w , while the outer cylinder is kept at the ambient temperature T_0 , such that $T_w > T_0$. The imposed temperature difference the cylinder walls generates buoyancy-induced natural convection flow along the vertical direction. The fluid thermal conductivity is assumed to vary linearly with temperature, while stochastic perturbations are introduced to represent random environmental and thermal fluctuations. The flow is assumed is assumed to be axisymmetric, laminar and fully developed, incompressible, one dimensional in the radial direction and governed by Boussinesq approximation.

The physical geometry of the problem together with the coordinate system, magnetic field direction, and thermal boundary conditions is illustrated in the figure below

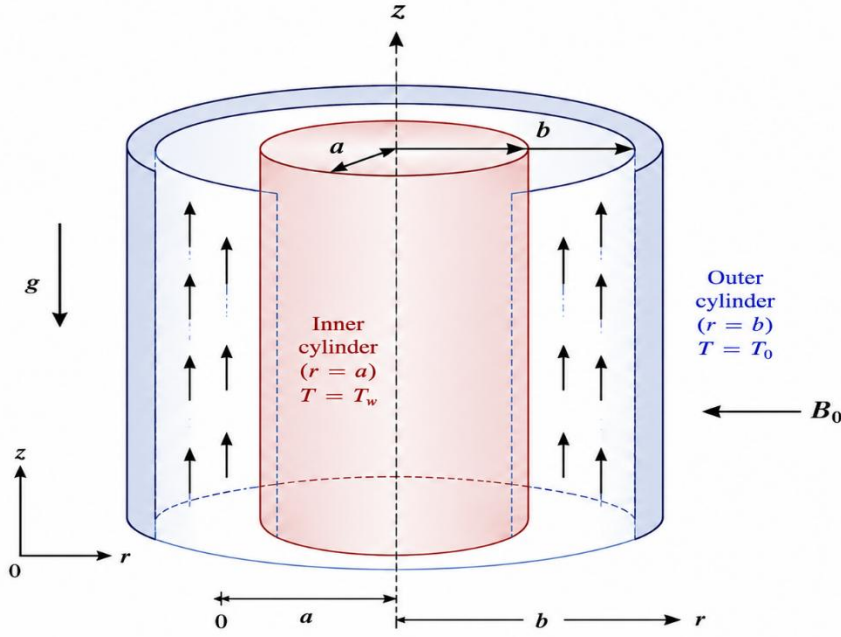


Figure 1: Geometry of the vertical annulus.

Let $u(r, t)$ denotes the axial velocity component and $u(r, t)$ represent the fluid temperature at radial coordinate r and time t .

In formulating the model, the following assumptions are considered:

1. The fluid is viscous, incompressible, and electrically conducting.
2. The flow is axisymmetric and fully developed.
3. The induced magnetic field is neglected due to the low magnetic Reynolds number.
4. The porous medium is homogeneous and isotropic.
5. Density variations are considered only in the buoyancy term according to the Boussinesq approximation.
6. Thermal conductivity varies linearly with temperature.
7. Random environmental effects are represented using Gaussian white noise processes.
8. Memory effects are incorporated through the Caputo–Fabrizio fractional derivative operator.
9. Viscous dissipation and Joule heating effects are neglected.
10. The cylinders are assumed to be infinitely long so that end effects are negligible.

The thermal conductivity of the fluid is assumed to depend linearly on temperature according to

$$k(T) = k_0[1 + \varepsilon(T - T_0)],$$

where:

k_0 is the reference thermal conductivity,

ε is the thermal conductivity variation parameter.

2.2. Governing Equations

Under the above assumptions, the dimensional governing equations for momentum and energy conservation are formulated as follows.

Momentum Equation

The fractional stochastic momentum equation governing the axial velocity distribution is given by

$$D_t^{\alpha,CF} u = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + g\beta(T - T_0) - \frac{\sigma B_0^2}{\rho} u - \frac{\nu}{K} u + \lambda_1 u \dot{W}(t) \quad (2.1)$$

where:

ρ denotes the fluid density, ν is the kinematic viscosity, g represents gravitational acceleration, β is the thermal expansion coefficient, σ denotes electrical conductivity, K is the permeability of the porous medium, λ_1 is the stochastic intensity parameter,

Energy Equation

The corresponding fractional stochastic energy equation is expressed as

$$D_t^{\alpha,CF} T = \frac{1}{\rho c_p} \frac{1}{r} \frac{\partial}{\partial r} \left(r k(T) \frac{\partial T}{\partial r} \right) + Q(T - T_0) + \lambda_2 T \dot{W}(t) \quad (2.2)$$

where;

c_p is the specific heat capacity, Q denotes the internal heat generation parameter, λ_2 represents the thermal stochastic intensity.

Initial and Boundary Conditions

Initially, the fluid is assumed to be stationary and maintained at ambient temperature:

$$u(r, 0) = 0, T(r, 0) = T_0.$$

The no-slip and thermal boundary conditions at the cylinder walls are prescribed as follows:

$$\text{At the inner cylinder } r = a: u(a, t) = 0, T(a, t) = T_w$$

$$\text{At the outer cylinder } r = b: u(b, t) = 0, T(b, t) = T_0.$$

2.3 Caputo–Fabrizio Fractional Operator

The Caputo–Fabrizio fractional derivative of order $0 < \alpha < 1$ is defined as follows:

$$D_t^{\alpha,CF} f(t) = \frac{1}{1-\alpha} \int_0^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \quad (2.3)$$

2.3. Dimensional Analysis

To obtain the dimensionless governing equations, the following variables are introduced:

$$R = \frac{r-a}{b-a}, \quad U = \frac{u}{U_0}, \quad \theta = \frac{T-T_0}{T_w-T_0} \text{ and dimensionless time variable: } \tau = \frac{\nu t}{(b-a)^2}.$$

Substituting the variables into equations (2.1) and (2.2) and processing it, the dimensionless equations governing the flow together with initial and boundary conditions are given as follows:

Momentum Equation

$$D_\tau^{\alpha,CF} U = \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + Gr \theta - Ha^2 U - \frac{1}{Da} U + \lambda_1 U \dot{W}(\tau) \quad (2.4)$$

Energy Equation

$$D_{\tau}^{\alpha,CF} \theta = \frac{1}{Pr} \frac{1}{R} \frac{\partial}{\partial R} \left[R(1 + \varepsilon \theta) \frac{\partial \theta}{\partial R} \right] + \Omega \theta + \lambda_2 \theta \dot{W}(\tau) \quad (2.5)$$

Initial conditions

$$U(R, 0) = 0, \theta(R, 0) = 0 \quad (2.6)$$

Boundary conditions

$$\begin{aligned} U(0, \tau) &= 0, \theta(0, \tau) = 1 \\ U(1, \tau) &= 0, \theta(1, \tau) = 0 \end{aligned} \quad (2.7)$$

Where;

Ha = Hartmann Number = $B_0(b-a)\sqrt{\frac{\sigma}{\mu}}$, Gr = Grashof Number = $\frac{g\beta(T_w - T_0)(b-a)^3}{\nu^2}$, Da = Darcy Number = $\frac{K}{(b-a)^2}$, Pr = Prandtl Number = $\frac{\mu c_p}{k_0}$ are the emerging controlling parameters and Ω denotes the dimensionless heat generation parameter.

3. Existence and Uniqueness of Solution

Momentum Equation

$$D_{\tau}^{\alpha,CF} U = \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + Gr \theta - Ha^2 U - \frac{1}{Da} U + \lambda_1 U \dot{W}(\tau) \quad (3.1)$$

Energy Equation

$$D_{\tau}^{\alpha,CF} \theta = \frac{1}{Pr} \frac{1}{R} \frac{\partial}{\partial R} \left[R(1 + \varepsilon \theta) \frac{\partial \theta}{\partial R} \right] + \Omega \theta + \lambda_2 \theta \dot{W}(\tau), \quad (3.2)$$

Initial conditions

$$U(R, 0) = 0, \theta(R, 0) = 0, \quad (3.3)$$

Boundary conditions

$$U(0, \tau) = 0, \theta(0, \tau) = 1, \quad (3.4)$$

$$U(1, \tau) = 0, \theta(1, \tau) = 0. \quad (3.5)$$

Equation (3.1) and (3.2) can be written as

$$D_{\tau}^{\alpha,CF} U = F(U, \Theta) \quad (3.6)$$

$$D_{\tau}^{\alpha,CF} \Theta = G(U, \Theta) \quad (3.7)$$

where

$$F(U, \Theta) = \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + Gr \Theta - Ha^2 U - \frac{1}{Da} U + \lambda_1 W_1(\tau) \quad (3.8)$$

and

$$G(U, \Theta) = \frac{1}{R} \frac{\partial}{\partial R} \left[R(1 + \varepsilon \Theta) \frac{\partial \Theta}{\partial R} \right] + Q\Theta + \lambda_2 W_2(\tau) \quad (3.9)$$

The operator F represents the combined effects of viscous diffusion, buoyancy force, magnetic damping, porous-medium resistance, and stochastic perturbations acting on the velocity field.

Similarly, the operator G represents nonlinear heat diffusion arising from temperature-dependent thermal conductivity, thermal generation effects, and stochastic thermal fluctuations.

The reformulation of the model into the operator equations above does not alter the physical content of the problem. Rather, it provides a compact mathematical representation that is suitable for the application of fixed-point theory. In particular, the abstract formulation enables the derivation of equivalent fractional integral equations through the Caputo-Fabrizio fractional integral operator. Once the corresponding solution operator is constructed, the existence of solutions can be established via Schauder's Fixed Point Theorem, while uniqueness follows from Banach's Contraction Principle under suitable Lipschitz conditions.

Furthermore, this operator representation forms the basis for investigating stability properties, continuous dependence on initial data, boundedness of solutions, and the computational admissibility of the model. Consequently, the abstract formulation serves as the fundamental mathematical framework upon which the theoretical analysis of the present fractional stochastic magnetoconvection problem is developed.

Theorem 4.1 (Well-Definedness of the Solution Operator)

Let

$$X = C([0, T]) \times C([0, T]) \quad (3.10)$$

be a Banach space equipped with the norm

$$\| (U, \Theta) \| = \max_{\tau \in [0, T]} |U(\tau)| + \max_{\tau \in [0, T]} |\Theta(\tau)| \quad (3.11)$$

Assume that:

1. The initial and boundary conditions are continuous.
2. The stochastic processes $W_1(\tau)$ and $W_2(\tau)$ are square-integrable on $[0, T]$.
3. The nonlinear conductivity term

$$(1 + \varepsilon \Theta) \frac{\partial \Theta}{\partial R}$$

is continuous and satisfies a Lipschitz condition with respect to Θ .

4. The parameters Ha , Da , Gr , ε , Ω are finite.

Then the integral operator

$$\mathcal{T} = (T_1, T_2) \quad (3.12)$$

defined by

$$T_1(U, \Theta) = U_0(R) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(U, \Theta) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^\tau F(U, \Theta) ds, \quad (3.13)$$

and

$$T_2(U, \Theta) = \Theta_0(R) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} G(U, \Theta) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^\tau G(U, \Theta) ds \quad (3.14)$$

is well-defined and maps bounded subsets of X into bounded subsets of X .

Proof

Using the Caputo–Fabrizio fractional integral operator, equations (3.1)–(3.2) can be written in the equivalent integral form

$$U(R, \tau) = U_0(R) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(U, \Theta) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^\tau F(U, \Theta) ds \quad (3.15)$$

and

$$\Theta(R, \tau) = \Theta_0(R) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} G(U, \Theta) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^\tau G(U, \Theta) ds \quad (3.16)$$

Define the operator

$$\mathcal{T}(U, \Theta) = (T_1(U, \Theta), T_2(U, \Theta)) \quad (3.17)$$

through the above integral equations.

Since the initial data are continuous, the parameters are finite, and the stochastic terms are square-integrable, both operators F and G are continuous on X . Furthermore, the Lipschitz continuity of the nonlinear conductivity term ensures that the nonlinear diffusion operator remains bounded on bounded subsets of X .

Therefore, for every bounded set $B \subset X$, there exists a constant $K > 0$ such that

$$\|\mathcal{T}(U, \Theta)\| \leq K, (U, \Theta) \in B.$$

Hence, the operator $\mathcal{T}$ is well-defined and maps bounded subsets of X into bounded subsets of X .

$$\mathcal{T}: X \rightarrow X$$

which completes the proof.

Theorem 4.2 (Existence and Uniqueness of Solution)

Assume the hypotheses of Theorem 4.1 hold. Furthermore, suppose there exists a constant $L > 0$ such that

$$\|F(U_1, \Theta_1) - F(U_2, \Theta_2)\| \leq L \|(U_1, \Theta_1) - (U_2, \Theta_2)\|,$$

and

$$\|G(U_1, \Theta_1) - G(U_2, \Theta_2)\| \leq L \|(U_1, \Theta_1) - (U_2, \Theta_2)\|.$$

Then the fractional stochastic MHD system (3.1)–(3.5) possesses a unique solution

$$(U, \Theta) \in X.$$

Lemma 4.1 (Boundedness of the Nonlinear Operators)

Let

$$X = C([0, T]) \times C([0, T]) \quad (3.18)$$

equipped with the norm

$$\| (U, \Theta) \| = \max_{\tau \in [0, T]} |U(\tau)| + \max_{\tau \in [0, T]} |\Theta(\tau)| \quad (3.19)$$

Assume that:

1. The stochastic processes $W_1(\tau)$ and $W_2(\tau)$ are bounded in mean square;
2. The parameters $Gr, Ha, Da, \varepsilon, \Omega$ are finite;
3. The functions U and Θ belong to a bounded subset $B_r \subset X$.

Then the operators

$$F(U, \Theta) = \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + Gr \Theta - Ha^2 U - \frac{1}{Da} U + \lambda_1 W_1(\tau) \quad (3.20)$$

and

$$G(U, \Theta) = \frac{1}{PrR} \frac{\partial}{\partial R} \left[R(1 + \varepsilon \Theta) \frac{\partial \Theta}{\partial R} \right] + \Omega \Theta + \lambda_2 W_2(\tau) \quad (3.21)$$

are bounded on B_r .

Proof

Since $U, \Theta \in B_r$, there exists a positive constant r such that

$$\| U \| \leq r, \quad \| \Theta \| \leq r.$$

Since all coefficients Gr, Ha, Da, ε , and Ω are finite, and the stochastic processes are bounded in mean square, there exist positive constants C_1 and C_2 satisfying

$$\| F(U, \Theta) \| \leq C_1,$$

and

$$\| G(U, \Theta) \| \leq C_2.$$

Therefore,

$$\max\{\| F(U, \Theta) \|, \| G(U, \Theta) \|\} \leq C,$$

where

$$C = \max\{C_1, C_2\} \quad (3.22)$$

Hence both operators are bounded on B_r .

$$\| F(U, \Theta) \| \leq C, \quad \| G(U, \Theta) \| \leq C$$

This completes the proof.

Theorem 4.3 (Boundedness of the Solution Operator)

Let $\mathcal{T}: X \rightarrow X$ be defined by

$$\mathcal{T}(U, \Theta) = (T_1(U, \Theta), T_2(U, \Theta)) \quad (3.23)$$

Where

$$T_1(U, \Theta) = U_0 + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(U, \Theta) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^\tau F(U, \Theta) ds \quad (3.24)$$

and

$$T_2(U, \Theta) = \Theta_0 + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} G(U, \Theta) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^\tau G(U, \Theta) ds \quad (3.25)$$

Then $\mathcal{T}$ maps a closed ball

$$B_r = \{(U, \Theta) \in X : \|(U, \Theta)\| \leq r\} \quad (3.26)$$

into itself.

Proof

From Lemma 4.1, there exists a constant $C > 0$ such that

$$\|F(U, \Theta)\| \leq C, \|G(U, \Theta)\| \leq C.$$

Therefore,

$$\|T_1(U, \Theta)\| \leq \|U_0\| + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} C + \frac{2\alpha T}{(2-\alpha)M(\alpha)} C$$

Similarly,

$$\|T_2(U, \Theta)\| \leq \|\Theta_0\| + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} C + \frac{2\alpha T}{(2-\alpha)M(\alpha)} C.$$

Combining the above estimates yields

$$\|\mathcal{T}(U, \Theta)\| \leq K,$$

where

$$K = \|U_0\| + \|\Theta_0\| + \frac{4(1-\alpha)}{(2-\alpha)M(\alpha)} C + \frac{4\alpha T}{(2-\alpha)M(\alpha)} C \quad (3.27)$$

Choosing $r \geq K$, we obtain

$$\|\mathcal{T}(U, \Theta)\| \leq r.$$

Hence

$$\mathcal{T}(B_r) \subseteq B_r.$$

Therefore, the operator $\mathcal{T}$ maps the closed ball B_r into itself.

$$\mathcal{T}(B_r) \subseteq B_r$$

This completes the proof.

Theorem 4.4 (Lipschitz Continuity of the Operators)

Let

$$X = C([0, T]) \times C([0, T]) \quad (3.28)$$

be endowed with the norm

$$\|(U, \Theta)\| = \max_{\tau \in [0, T]} |U(\tau)| + \max_{\tau \in [0, T]} |\Theta(\tau)|.$$

Assume that:

1. The parameters Gr , Ha , Da , ε , Pr , and Ω are finite;
2. The stochastic processes $W_1(\tau)$ and $W_2(\tau)$ are bounded;
3. The nonlinear thermal conductivity term

$$(1 + \varepsilon\Theta) \frac{\partial \Theta}{\partial R}$$

satisfies a Lipschitz condition with respect to Θ .

Then the operators F and G defined by

$$F(U, \Theta) = \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + Gr \Theta - Ha^2 U - \frac{1}{Da} U + \lambda_1 W_1(\tau) \quad (3.29)$$

and

$$G(U, \Theta) = \frac{1}{PrR} \frac{\partial}{\partial R} \left[R(1 + \varepsilon\Theta) \frac{\partial \Theta}{\partial R} \right] + \Omega \Theta + \lambda_2 W_2(\tau) \quad (3.30)$$

are Lipschitz continuous on X .

Proof

Let

$$(U_1, \Theta_1), (U_2, \Theta_2) \in X.$$

Consider the difference

$$F(U_1, \Theta_1) - F(U_2, \Theta_2).$$

Using the definition of F ,

$$F(U_1, \Theta_1) - F(U_2, \Theta_2) = \frac{\partial^2(U_1 - U_2)}{\partial R^2} + \frac{1}{R} \frac{\partial(U_1 - U_2)}{\partial R} + Gr(\Theta_1 - \Theta_2) - \left(Ha^2 + \frac{1}{Da}\right)(U_1 - U_2) \quad (3.31)$$

Taking norms and applying the triangle inequality gives

$$\|F(U_1, \Theta_1) - F(U_2, \Theta_2)\| \leq L_1(\|U_1 - U_2\| + \|\Theta_1 - \Theta_2\|) \quad (3.32)$$

where $L_1 > 0$ is a constant depending on

$$Ha, Da, Gr.$$

Hence F is Lipschitz continuous.

Similarly,

$$G(U_1, \Theta_1) - G(U_2, \Theta_2) = \frac{1}{PrR} \frac{\partial}{\partial R} \left[R \left((1 + \varepsilon\Theta_1) \frac{\partial\Theta_1}{\partial R} - (1 + \varepsilon\Theta_2) \frac{\partial\Theta_2}{\partial R} \right) \right] + \Omega(\Theta_1 - \Theta_2) \quad (3.33)$$

Since the nonlinear conductivity term is assumed to satisfy a Lipschitz condition, there exists a constant $L_2 > 0$ such that

$$\|G(U_1, \Theta_1) - G(U_2, \Theta_2)\| \leq L_2(\|U_1 - U_2\| + \|\Theta_1 - \Theta_2\|).$$

Therefore, both operators F and G are Lipschitz continuous on X .

$$\|F(U_1, \Theta_1) - F(U_2, \Theta_2)\| \leq L_1 \| (U_1, \Theta_1) - (U_2, \Theta_2) \|$$

And

$$\|G(U_1, \Theta_1) - G(U_2, \Theta_2)\| \leq L_2 \| (U_1, \Theta_1) - (U_2, \Theta_2) \|.$$

This completes the proof.

Corollary 4.1 (Lipschitz Continuity of the Solution Operator)

Let $\mathcal{T}: X \rightarrow X$ be the solution operator defined by the Caputo–Fabrizio integral equations. Then $\mathcal{T}$ satisfies

$$\|\mathcal{T}(U_1, \Theta_1) - \mathcal{T}(U_2, \Theta_2)\| \leq L \| (U_1, \Theta_1) - (U_2, \Theta_2) \|,$$

where

$$L = \frac{2(1-\alpha)+2\alpha T}{(2-\alpha)M(\alpha)} \max\{L_1, L_2\} \quad (3.34)$$

Proof

Using the definition of $\mathcal{T}$ and the Lipschitz estimates established above,

$$\begin{aligned} \|\mathcal{T}(U_1, \Theta_1) - \mathcal{T}(U_2, \Theta_2)\| &\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \max(L_1, L_2) \|(U_1, \Theta_1) - (U_2, \Theta_2)\| \\ &+ \frac{2\alpha T}{(2-\alpha)M(\alpha)} \max(L_1, L_2) \|(U_1, \Theta_1) - (U_2, \Theta_2)\| \end{aligned} \quad (3.35)$$

Combining terms yields

$$\|\mathcal{T}(U_1, \Theta_1) - \mathcal{T}(U_2, \Theta_2)\| \leq L \|(U_1, \Theta_1) - (U_2, \Theta_2)\|,$$

where

$$L = \frac{2(1-\alpha) + 2\alpha T}{(2-\alpha)M(\alpha)} \max(L_1, L_2) \quad (3.36)$$

Hence $\mathcal{T}$ is Lipschitz continuous

$$\|\mathcal{T}(U_1, \Theta_1) - \mathcal{T}(U_2, \Theta_2)\| \leq L \|(U_1, \Theta_1) - (U_2, \Theta_2)\|.$$

This completes the proof.

Theorem 4.5 : (Contraction Property of the Solution Operator)

Let

$$\mathcal{T}: X \rightarrow X$$

be the solution operator associated with the fractional stochastic MHD system (3.1)–(3.5).

Assume that the hypotheses of Theorem 4.4 hold and that

$$L = \frac{2(1-\alpha) + 2\alpha T}{(2-\alpha)M(\alpha)} \max(L_1, L_2) < 1 \quad (3.37)$$

Then the operator $\mathcal{T}$ is a contraction mapping on X .

Proof

From Corollary 4.1, for any two elements

$$(U_1, \Theta_1), (U_2, \Theta_2) \in X,$$

we have

$$\|\mathcal{T}(U_1, \Theta_1) - \mathcal{T}(U_2, \Theta_2)\| \leq L \|(U_1, \Theta_1) - (U_2, \Theta_2)\|.$$

Since $L < 1$, it follows immediately that

$$\|\mathcal{T}(U_1, \Theta_1) - \mathcal{T}(U_2, \Theta_2)\| < \|(U_1, \Theta_1) - (U_2, \Theta_2)\|.$$

Hence, $\mathcal{T}$ reduces the distance between any two points in the Banach space X .

Therefore, $\mathcal{T}$ satisfies the definition of a contraction mapping

$$\mathcal{T} \text{ is a contraction on } X$$

This completes the proof.

Theorem 4.6 (Existence and Uniqueness of Solution)

Assume that:

1. The hypotheses of Theorems 4.1–4.5 are satisfied;
2. The solution operator

$$\mathcal{T}: X \rightarrow X$$

maps a closed ball $B_r \subset X$ into itself;

3. The contraction condition

$$L < 1$$

holds.

Then the fractional stochastic magnetoconvection system (3.1)–(3.5) admits a unique solution

$$(U, \Theta) \in X.$$

Proof

By Theorem 4.3, the operator $\mathcal{T}$ maps the closed ball

$$B_r = \{(U, \Theta) \in X : \|(U, \Theta)\| \leq r\} \quad (3.38)$$

into itself.

Furthermore, by Theorem 4.5, $\mathcal{T}$ is a contraction mapping on B_r .

Since

$$X = C([0, T]) \times C([0, T]) \quad (3.39)$$

is a Banach space, every closed subset of X , including B_r , is complete.

Therefore, all the hypotheses of the Banach Fixed Point Theorem are satisfied.

Consequently, there exists a unique element

$$(U, \Theta) \in B_r$$

such that

$$\mathcal{T}(U, \Theta) = (U, \Theta) \quad (3.40)$$

The fixed point of $\mathcal{T}$ is precisely the solution of the equivalent integral formulation of the fractional stochastic MHD system and hence satisfies equations (3.1)–(3.5).

Therefore, the model possesses a unique solution on the interval

$$[0, T]$$

$$(U, \Theta) \in C([0, T]) \times C([0, T])$$

and the solution is unique.

This completes the proof.

Remark 4.1 (Computational Well-Posedness)

The existence and uniqueness results established above imply that the proposed fractional stochastic magnetoconvection model is well-posed in the sense of Hadamard. Specifically:

1. A solution exists.
2. The solution is unique.
3. The solution depends continuously on the model parameters and initial data.

Consequently, numerical approximations generated by finite difference, finite element, spectral, collocation, or other computational schemes converge to the same physical solution. Therefore, the model is mathematically admissible for computational implementation and numerical simulation.

The model is computationally well-posed

Apply Banach Fixed Point Theorem

Since

1. X is a complete Banach space,
2. $\mathcal{T}$ is a contraction,

Banach's theorem guarantees a unique fixed point

$$(U, \Theta) = \mathcal{T}(U, \Theta) \tag{3.41}$$

Therefore the fractional stochastic MHD system admits a unique solution

The solution (U, Θ) exists and is unique

5. Conclusion

In this study, a fractional stochastic magnetoconvection model describing natural convection flow of an electrically conducting fluid in a porous vertical annulus with variable thermal conductivity was investigated. The governing equations were formulated using the Caputo–Fabrizio fractional derivative in order to incorporate memory effects, while stochastic perturbations were included to account for random environmental and thermal fluctuations.

To establish the mathematical validity of the proposed model, the governing fractional stochastic partial differential equations were reformulated as an equivalent system of fractional integral equations. An appropriate solution operator was constructed in a suitable Banach space, and its fundamental properties were investigated. It was shown that the associated nonlinear operators are bounded and satisfy the required Lipschitz conditions. Consequently, the solution operator maps a closed ball into itself and satisfies the contraction property under suitable assumptions on the model parameters.

By applying the Banach Fixed Point Theorem, the existence and uniqueness of solutions to the fractional stochastic magnetoconvection system were established. These results guarantee that the proposed model admits a unique physically meaningful solution for the prescribed initial and boundary conditions. Furthermore, the uniqueness result ensures that repeated computations performed under identical conditions will converge to the same solution.

The theoretical analysis further demonstrates that the model is well-posed in the sense of Hadamard, implying existence, uniqueness, and continuous dependence of solutions on the initial data and governing parameters. Consequently, the proposed fractional stochastic MHD model is mathematically admissible for computational implementation and provides a reliable framework for future numerical investigations.

The results obtained in this work provide a rigorous theoretical foundation for the study of fractional stochastic magnetoconvection phenomena in porous annular geometries

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